gaṇa found by us is what is called Madhyama Sāvana-ahargaṇa or number of days according to mean solar reckoning, a reckoning made on the basis of taking uniform motion of the Sun in right ascension. The planets computed for the Sun-rise on this basis, are to be corrected for the true Sun-rise which goes according to the measure of ☉ ie. the true longitude of the Sun. But in the name of Bhujāntara we have effected the correction for $\odot - l$; so what remains is to effect correction for $l - \alpha$. This correction is known as Udayāntara.

In other words, since we could not compute the Sphuta Sāvana-ahargaṇa, when we deem that x days have elapsed according to the mean solar reckoning from the epoch upto the mean Sun-rise $x \pm f$ might have elapsed according to Sphuta Sāvana-ahargaṇa where f is a fraction. A correction is to be effected for this fraction f of a day and it is of the form $\odot - \alpha = \odot - l + l - \alpha$. In the beginning Bhāskara said that mean planets are had being computed out of the mean solar ahargaṇa, at the time when the mean Sun is about the eastern horizon of the equator. Why did he say 'about the horizon'? It is because the mean solar ahargaṇa indicates a Sun-rise on the basis of equal motion in right ascension whereas on the basis of equal motion in l, he would be about the horizon. Had we been able to compute Sphuta-Sāvana-ahargaṇa, we could have got direct the planetary position when the true Sun is on the horizon. The correction for the difference in ☉ and l having been attended to through Bhujāntara, we are now to attend to the correction for the difference $l - \alpha$.

Verse 64. Another way of looking at the same.

Had we obtained the Ahargaṇa in terms of the local risings of the Rasis in the place of the equatorial, and computed the planetary positions for the local true Sun-rise obtained that way, we would have done the three corrections namely Bhujāntara, Chara and the Udayāntara as well.

Comm. The correction of chara arises out of the difference between the equatorial risings and local risings of the mean Sun. The local mean Sun pertaing again to uniform motion along the celestial equator Udayāntara has to be effected for $l-\alpha$, ie. for the local mean Sun on the ecliptic. Thus we have obtained the planetary positions for the local mean Sun-rise and to obtain them for the True Sun-rise, Bhujāntara is to be effected.

Verse 65. An alternative method of effecting the Udayāntara correction.

Double the H sine of the Sāyana longitude of the Sun derived out of the smaller H sin-table, being multiplied by the daily motion of the planet and divided by 270 and the result in seconds of arc is to be corrected in the planetary position positive or negative according as the Sun is in the even or odd quadrants.

Comm. (1) In symbols the correction indicated is $\frac{m \text{ H sin } 2l}{270}$ where m is the planet's mean daily motion in minutes of arc, l the longitude of the mean Sun, and H sine is taken where the radius$=120$. In the commentary under the verse, Bhāskara adds 'Each quadrant of the eclipse rises (at the equator) in $\frac{1}{4}$th of a day but each Rāśi does not rise in $\frac{1}{12}$ of a day. Since this Udayāntara correction vanishes when the Sun is at the ends of quadrants it must be construed that this correction increases positively or negatively from the beginning of the quadrant to the middle and decreases from the middle to the end of a quadrant. Saying that a particular Rāśi rises at the equator in n nādikās is only an approximate statement since the Rāśi does not rise uniformly. That is why astronomers like Aryabhata stipulated finding the risings of smaller arcs like horas and Dṛkkāṇas. Find $\frac{H \sin l \times H \cos \omega}{H \cos \delta}$ ie. H sin α. Take the asus in the arc of this H sine ie. find α in minutes. Then $l-\alpha$ gives the number of asus for which the correction in the planetary position is to be effected, for, by these asus the True Sun-rise is accelerated or belated. In the middle of a quadrant these asus will be a little above 26 Vinadis. To obtain them at any point of the quadrant, take H sin $2l$ as the argument so that the maximum correction will be had at the middle of the quadrant by this argument. Then the rule of three is 'If by 120 as radius, we have 26 Vinadis, what shall we have for H sin $2l$?' The result is $\frac{H \sin 2l \times 26}{120} = \frac{H \sin 2l}{4^1}$ approximately. Then another rule of three. "If by 60 Vinadis we have m' of the planetary motion, where the daily motion of the planet is $m°$, what shall we have for $\frac{H \sin 2l}{4\frac{1}{2}}$?" The result is $\frac{H \sin 2l \times m}{60 \times 4\frac{1}{2}}$ minutes $= \frac{H \sin 2l \times m}{270}$. Then the sign of the correction is clear.

(2) We shall now prove Bhāskara's statement that at the middle of the quadrant, the correction is 26 Vinadis. We saw above that $l-\alpha = \tan^2 \omega/2$ H sin $2l$. It is really creditable on the part of Bhāskara to have seen by intuition that the argument is H sin $2l$. The maximum correction is therefore $\tan^2 \omega/2$ expressed in asus or minutes of arc. Let $x = \tan^2 \omega/2$

$\log x = 2 \log \tan \omega/2$. Take $\omega = 24$

Then $\log x = 2 \times \bar{1}.3275 = \bar{2}.6550$

$\therefore$ $x =$ ·04519 radian = 155 minutes of arc ie. 155 asus $= \frac{155}{6} = 26$ apply.

(3) Taking the case of the Moon, the max correction amounts to $\frac{790 \times 120}{270} = 5' - 51''$.

66, 67. The computation of Tithi, Nakṣatra and Yoga.

The elongation of the Moon ie. the excess of the Moon's longitude over that of the Sun in degrees. (In case the former is smaller, add 360° to it and subtract Sun's longitude) being divided by 12 and 6, the quotients represent the elapsed tithis and Karaṇas. If the elapsed Karanas are K, count K − 1 beginning from Bava to get the current Karaṇa and count from S'akuni to get the current one beginning from the mid-moment of Kṛṣnachaturdasī. Take again the planetary position or that of the Moon in particular as well as the sum of the longitudes of the Sun and the Moon both expressed in minutes of arc and divided by 800. The first quotient gives the elapsed stars ie. the Stars covered by the planet or the Moon; whereas the second quotient gives the elapsed Karaṇas. Then take the remainders in seconds and divide by the respective daily motions in minutes i.e. in the case of the planets or the Moon divide by their respective motions and in the case of yogas divide by the sum of the motions of the Sun and the Moon. Then the results give the times in nādīs as to how much the next nakṣatra or yoga have elapsed. If it be required to find as to how long the next nakṣatra or yoga will last, substract the remainder from 800, and divide by the daily motions in minutes as mentioned above. The results give as to how many nādis beginning from the morning concerned, the next nakṣatra or yoga will last.

Comm. (1) A lunation i.e. the time from the moment of New Moon to the next New Moon is divided into 30 parts called Tithis. Their names are pratipat, Dwitīyā etc. upto the 15th purnima or full Moon and again pratipat, Dwitīyā etc. upto the 30th i.e. Amāvāsyā or New Moon. Thus pratipat starts when ☾ = ⊙ i.e. when the longitude of the Moon is equal to that of the Sun i.e. from the moment of New Moon when the Moon is in conjunction with the Sun. अमा सह=वसतः सूर्याचन्द्रमसावस्या मित्यमावस्या i.e. Amāvāsyā is that point of time when the Sun and the Moon are together. Pratipat lasts till ☾ = ⊙ + 12°; then Dwitīyā begins and lasts till ☾ = ⊙ + 24; Thus purnima begins from the moment when ☾ = ⊙ + 168 and lasts till ☾ = ⊙ + 180° and Amāvasyā begins when ☾ = ⊙ + 348 and lasts till ☾ is again equal to ⊙ Thus a tithi is the time that is taken by the Moon to overcome the Sun by 12° beginning from the moment of New Moon. In other words the tithi is a measure of the phase of the Moon with a particular convention.

(2) In modern astronomy the phase of the Moon or that of a planet is measured by the formula $\frac{1 + \cos EPS}{2}$ where P is the planet or the Moon. Since ES is almost

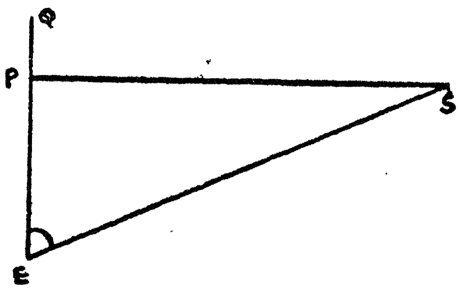


Fig. 29

parallel to PS (Fig. 29) $\widehat{EPS}$ may be taken to be nearly equal to $\widehat{PES}$ which is the elogation of the planet or the Moon so that phase $= \frac{1 - \cos EPS}{2}$ approximately. So in modern astronomy also the phase is measured through the elongation. But the maximum phase in modern astronomy is taken to be unity as could be seen from the formula, for, when $\widehat{EPS} = 180°$, the phase $= 1$. Thus the phase multiplied by 15 gives approximately the tithi. We shall have occasion to deal with this topic later.

(3) Suppose ☾ − ☉ = E°. The number of the tithis elapsed is equal to the quotient in $\frac{E}{12}$. Take the remainder $r°$. It means during the current tithi which has a duration of 12°, $r°$ have elapsed. Let the daily motions of the Moon and the Sun be m and s so that the Moon overtakes the Sun by $m - s$ (expressed in minutes for convenience) during 60 nādis. The elapsed time of the current tithi in nādis is given by $\frac{r \times 60 \times 60}{(m - s)}$. But $r \times 60 \times 60$ = Seconds of arc of the remainder so that it is said "गतैष्यविलिप्तिकाः". If it be required to find how many nādīs the next tithi lasts, $(12-r)°$ or $(12-r) \times 60'$ is to be gained by the Moon over the Sun. So the time in

$$\text{Nādīs} = \frac{(12 - r) \times 60 \times 60}{m - s} = \frac{\text{एष्यविलिप्तिकाः}}{m - s}.$$

(4) A lunation is again divided into 60 Karaṇas and so 6° of increase in elongation correspond to a Karaṇa. These Karaṇas are eleven in number, out of which 4 are fixed to occur during the latter half of Kṛṣna Chauturdas'ī, during the two halves of Amāvāsyā and the first half of the S'ukla pratipat. Their names are S'akuni, Chatuṣpāt, Nāgava, and Kimstughna. So there remain 56 halves of tithis during the lunation during which the remaining seven Karaṇas, Known as Bava, Bālava etc. rotate eight times. Thus the first half of the duration of any tithi is covered by one Karaṇa and the latter by another. The computations of Karaṇas proceeds as with respect to tithis

but dividing the elongation by 6, we are asked to count K − 1, (where K is the quotient) beginning with Bava because the first half of S'ukla pratipat is covered by the fixed Karaṇa Kimstughna. The computation of the elapsed as well as the remaining nādīkas of a particular Karaṇa, it proceeds on the same lines as that of a tithi.

(5) The Nakṣatra in which a planet or the Moon is situated is then found. The zodiac is divided into 27 equal divisions beginning with its zero point and each division is named after the brilliant star of that division. Those stars are As'winī, Bharaṇī etc. The star occupied by the Moon has a special significance in the Hindu calendar, it being spoken of that every day is presided over by a nakṣatra. This happens so because the Moon's sidereal period is approximately 27 days.

(6) To compute the Nakṣatra in which the planet or Moon is, take its longitude in minutes of arc λ; divide by 800, since each star division consists of $\frac{360^\circ}{27} = \frac{360 \times 60}{27}$ minutes $= 200'$. The quotient gives the elapsed nakṣatras. To get the elapsed nādikās of the current nakṣatra or the remaining, let the remainder be r'. Then the proportion is 'If during the day of 60 nādis, the planet or the Moon goes p minutes what time does it take to cover r' or $800-r'$. The answer is

$$\frac{r' \times 60}{p'} \text{ or } \frac{(800-r') \times 60}{p'} - \qquad p' \quad .$$

(7) Regarding yogas, let the longitudes of the Sun and the Moon be $\odot$ and λ in minutes of arc; let their daily motions be s and m in minutes. If the sum of the longitudes is $800'$, we say the first yoga named Viṣkambha is over, if the sum is $1600'$, the second Yoga, named prīti has elapsed. Thus going on if the sum is 360°, we say the 27 yogas have elapsed and the first again begins. Since the sum of the daily motions of the Sun and the Moon

are on the average $59'-8''+790'-35''=850'$, to cover one Yoga, they take roughly one day since the duration of a yoga is of 800′. To get the number of elapsed yogas, divide $(\odot+M)'$ where M is the longitude of the Moon by $(s+m)$. To get the elapsed nādīs which are given by the remainder r' or the remaining nādīs of the current yoga which are given by 800; the proportion is 'If by $s+m$ gain in the sum of the longitudes we have 60 nādīs, what time is indicated by r' or $800-r'$?'. The answer is

$$\frac{r' \times 60}{m+s} \text{ or } \frac{(800-r)' \times 60}{m+s} = \frac{\text{गतैष्यविलिप्तिकाः}}{m+s}.$$

By the word गतैष्यविलिप्तिकाः is meant therefore the number of seconds of arc as many as the remainder r or $(800-r)$ is in minutes or what is the same the remainder or $800-r$ converted into seconds.

(8) The tithi, karaṇa, nakṣatra of the Moon and yoga constitute four of the Angas of the panchānga or the Hindu Calendar the fifth being the week-day. All these five are supposed to have their effects good or bad on living beings.

Verses 68, 69. The correction what is called Natakarma.

The Zenith distances of the Sun and the Moon at the end of Purṇimā or Amāvāsyā at the time of lunar or solar eclipse, being expressed in nādis, are multiplied by 6 to get the degrees. Let their H sine be got from the short table of H sines. Multiply it by the equations of centre of the Sun and the Moon. Divide by 4920, and 4361 respectively. If the Sun be in the Eastern hemisphere, let the result pertaining to the Sun be subtracted from his position; if in the Western, let it be added to his position. If the Moon be in the Eastern hemisphere and if his equation of centre be negative let the result be added to his position; if the equation of centre be positive, let the result be subtracted from his position in either of the hemispheres.

Again from these positions, let the tithi be computed and again let the above process be carried out until a constant time is arrived at for conjunction or opposition.

Comm. (1) The above procedure is accepted by Bhaskara as Āgama enunciated by Brahmagupta and reiterated by Chaturveda as giving results that accorded with observation. We shall see that the correction known in modern astronomy as 'correction due to astronomical refraction' is indicated here, though it was not stated explicitly.

In fact what was stated by Brahmagupta was that the periphery of the Manda epicycle given as $13\frac{2}{3}°$ for the Sun and as 31–36 for the Moon hold good only on the meridian but the periphery of the Sun is to be increased or decreased by 20′ according as the equation of centre is negative and the Sun is the Eastern equatorial horizon, or western equatorial horizon. If the equation of centre be positive the reverse correction is to be effected in the periphery i.e. for negative equation of centre.

Periphery on the Eastern equatorial horizon = 14°–0
On the meridian = 13°–40′
On the Western equatorial horizon = 13°–20′
For positive equation of centre
Periphery on the Eastern unmandala i.e.
equatorial horizon = 13°–20′
On the meridian = 13°–40′
On the Western unmandala = 14°–0

In the case of the Moon, for negative equation of centre.

On the Eastern unmandala = 30 – 44
On the meridian = 31 – 36
On the Western unmandala = 32 – 28

For positive Equation of centre.

On the Eastern unmandala	= 30 – 44
On the meridian	= 31 – 36
On the Western unmandala	= 30 – 44

In between the meridian and the unmandala, proportion is to be used. If by H sine Z equal to R, there is a difference of 20′ in the periphery of the Sun, what will it be for an arbitrary H sin Z ? The result is $\text{H sin Z} \times \frac{1}{3} \times \frac{1}{120} = \frac{\text{H sin Z}}{360}$. Then again another proportion "If by $13\frac{2}{3}°$ we have the equation of centre E_1 what shall we have for the above difference ?"

$$\text{The result is } \frac{\text{H sin Z}}{360} \times \frac{E_1 \times 3}{41} = \frac{E_1 \text{ H sin Z}}{4920}$$

$$\text{Similarly for the Moon } - \frac{E_2 \text{ H sin Z} \times 52}{60 \times 120 \times \text{[illegible]}}$$

$$= \frac{E_2 \times \text{H sin Z} \times 52 \times 5}{60 \times 120 \times 158} = \frac{E_2 \text{ H sin Z}}{56880/13} = \frac{E_2 \text{ H sin Z}}{4376}$$

which is taken as 4361 on account of approximating $\frac{52}{60 \times 120} = \frac{1}{138}$ and then multiplying by $\frac{5}{158}$

$$= \frac{1}{138} \times \frac{1}{158} = \frac{5}{21804} = \frac{1}{2180 \times \frac{4}{5}} = \frac{1}{4361}.$$

(2) In modern astronomy on account of astronomical refraction a celestial body is elevated towards the zenith by the formula K·tan Z where K has a particular value for all celestial bodies. Thus a body in the Eastern hemisphere, getting elevated the correction is to be negative and in the Western it is to be positive. We shall see how far the given correction accords with this.

(3) Consider for the Sun first for negative equation of centre.

(a) On the east, the negative equation being increased, elevation is effected.

(b) On the west, the negative equation of centre being rendered less, elevation is effected.

(c) For positive equation of centre, on the east it being lessened, again elevation is effected.

(d) For +ve equation of centre on the Western side it being increased again it is elevated.

Thus with respect to the Sun, the stipulated correction effects elevation in all the cases which accords with the effect of the modern refraction.

Then let us consider the case of the Moon.

(e) In the case of negative equation of centre, it being lessened in the East, the effect is depression.

(f) And being increased in the west, the effect is depression.

(g) In the case of positive equation of centre, it being reduced in the east, elevation is effected.

(h) And in the west, it being reduced, elevation is effected.

Thus in the case of the Moon, the phenomenon is recorded as depression in the case of negative equation of centre. Out of these two cases again the equation of centre being lessened is truly desirable as the Hindu equation of centre is in excess of the true value. So it need not be interpreted as depression. Regarding the other case, the error might have been due to the fact that a lesser parallax being taken, whose effect is to depress the celestial body, depression might have been noticed during the course of an eclipse wherein conjunction or opposition had to be belated; or again the greater equation of centre as was postulated for the Sun, than what it should be when his equation of centre was negative might have depressed the Sun, so that the Moon had to be depressed to arrive at the correct moment of conjunction or opposition. Thus this correction of Natakarma which was accepted by Bhāskara on the reported Āgama of Brahmagupta and also on the right endorsement of Chaturveda, must have been in fact no other than the effect of the phenomenon of astronomical refraction, and what further strengthens this observation is the prescription of H sin Z which is proportional tan Z. Also the modern formula being A tan Z where $A = 58''$ approximately, when Z is sufficiently large, the magnitudes given by Brahmagupta are of the same order as that of the moderns. This correction of Natakarma really reflects much credit on the ancient Hindu observations.

Verse 70. Computing the planetary position for a given moment.

The daily motion of the planet being multiplied by the time that has elapsed or that is to elapse at which the planetary position is to be found, and divided by 60, and the result being substracted from or added to the planetary position found, will render the position hold good for the moment in question. The Sun and the Moon will become by this process of what is called तात्कालिकीव equal up to minutes for the moment of conjunction or opposition. For opposition only the Rāśis differ whereas the degrees, minutes and seconds in their positions will be equal whereas for opposition, the positions are equal in all respects ie. Rāśis, degrees, minutes and seconds too.

Comm. The meaning is clear.

Verses 71 to 75. Obtaining what are called Suksma-nakṣatras.

The computation of the nakṣatras done as prescribed before, is only approximate. Now I shall give the method of obtaining what are called Sukṣma-nakṣatras as prescribed by the Riṣis that are required to note auspicious occasions regarding marriages, journeys etc. people who knew about it, told that the six stars Vis'ākha, Punarvasu, Rohiṇī, and the three Uttaras or Uttaraphalgunī, Uttarāṣādhā, Uttarabhadrā have the duration of one and half stars ie. $3/2 \times 790' - 35'' = 1185' - 52''$. The six stars Āslesha, Ārdrā, Swātī, Bharaṇī, Jyeshthā and S'atabhishak have half the duration of a star ie. $395' - 17''$. The remaining 15 alone have one nakṣatra duration ie. $790' - 35''$. A star's duration is the mean daily motion of the Moon ie. $790' - 35''$. The sum total of all the above 27 stars being subtracted from 360°, give the duration of the star what is called Abhijit which occurs after Uttarāṣādha and before S'ravaṇa. To obtain the star in which a planet is situated, convert its longitude in minutes of arc and substract the durations of the stars from As'wini as many as could be substracted. The number of stars whose durations are thus substracted are deemed to have elapsed. The remainder is called the gata or elapsed portion of the current star and the difference of this gata and the duration of the current nakṣatra is called the Ēṣya, ie. unelapsed portion. To obtain the elapsed time or the unelapsed time of the current star the gata or the ēṣya is to be multiplied by 60, and divided by the daily motion of the planet concerned the result being in nādīs.

Comm. One line in the verse 72 is evidently missing which should name Rohiṇī and the three Uttaras. We are able to know them from Bhāskara's commentary as well as from Brahmagupta and S'rīpati. In the course of the commentary Bhāskara reiterates what was stated by Brahmagupta, that Risis like Pulis'a, Vasistha and Garga spoke about these Suksmanaksatras. The duration of Abhijit calculated as directed is 254′ – 18″. The computation as directed is easy for understanding. The reason for the durations indicated is not clear but is to be taken as based on Astrology.

Verses 76, 77. The duration of the planets' transit into new Rās'is and the duration of the interval between successive stars, tithis, Karaṇas and yogas.

The disc of the planet multiplied by 60, and divided by its daily motion gives the nādīs of transit of the planet from Rās'i to Rās'i. This duration is considered to be holy for performing Vedic rites. It is the holiest with respect to the Sun's transit in particular. A planet in its transit gives partly holy results not so much as the Sun, depending upon the nature of the previous and succeeding Rās'is. The duration of Sandhi for tithis is got by dividing the disc of the Moon expressed in seconds by the difference of the daily motions of the Moon and the Sun; so also with respect to Karaṇas. The Sandhi between two Nakṣatras is obtained by the same measure of the disc of the Moon expressed in seconds of arc being divided by the Moon's daily motion. The Sandhi between two yogas is got by dividing the same numerator by the sum of the daily motions of the Sun and the Moon.

Comm. (1) The Sandhi is the period that elapses during the transit of the disc concerned between the Rās'is and nakṣatra divisions. With respect to tithi, Karaṇa and yoga, the divisions are imaginary not being seen in the Sky and the disc concerned is that of the Moon, and not that of the Sun though the Sun's motion is also taken into account. We say a transit from a division to another is current so long as the disc lies partly in the previous and partly in the latter. So the Sandhi begins when the disc touches the next division and ends when its hind part touches the previous division. In other words it is the interval between the first contact of the disc with the succeeding division and the last contact with the previous division.

(2) In the case of the Sun the duration of the Sanskrānti is equal to $\frac{32'\frac{1}{2}}{59-2}$ of a day taking the maximum magnitude of the disc and the minimum daily motion approximately $= \frac{65}{2 \times 57} \times 60$ nādīs $= \frac{650}{19} = 34 - 12$ nādīs approximately or more accurately $\frac{32\frac{21}{40}}{56\frac{11}{12}}$

$$= \frac{1301}{40} \times \frac{12}{683} = \frac{3903}{6830} \text{ of a day}$$

$$= \frac{3903}{683} \times 6 \text{ nādīs} = \frac{23418}{683} = 34-17 \text{ nādikās.}$$

Taking liberal boundaries, the वृद्धकारिका or the elders saying is विंशतिः पूर्वे, विंशतिः परे ie. 40 nādikās on the whole.

Here ends the Spaṣtāshikāra.

ॐ

THE TRIPRAS'NĀDHIKĀRA

Verse 1. The purport of this chapter.

Pandits say that this is the science of time in as much as, herein there is described the method of knowing the direction and the point of space (where a celestial body is situated) given the time. Hence I expound that chapter, which gives that knowledge and which abounds in very important statements, which forms the quintessence of the science of astronomy.

Comm. This chapter is called Tripras'nādhikāra, since this deals with the three questions pertaining to the direction and the point of space of a celestial body for a given time ie. dealing with Des'a, Dik and Kāla. In this chapter we come across the Hindu methods of spherical trigonometry, and gnomonics or S'ankuvedha or observations with the help of a gnomon. Also we find herein a usage of what is called 'Golayantra' or the armillary sphere, which helped the Hindu astronomers to solve all diurnal problems. We find herein Bhāskara excelling himself. This chapter abounds in a good number of technical terms and without a knowledge of this chapter, no one could call himself a Hindu astronomer.

Verses 2–4. To compute what is called lagna given the time.

The lagna or the ascendant as it is called or the point of intersection of the ecliptic with the horizon at a given point of time is obtained as follows. Obtain the Sāyana longitude of the Sun at the point of time at which it is required to find the lagna. Supposing the Sun is in the rth degree of a particular Rās'i, the number of asus which give the rising time of the arc of $(30 - r)^\circ$ of that Rās'i are called the Bhogyāsus, or the asus which are taken by the remainder of the Rās'i to rise at the place. They are equal to. $\frac{(30 - r) \times T}{30}$ where T gives the asus of the rising time of that Rās'i. The Bhuktāsus on the other hand are the asus which pertain to the rising time of the arc of the first r° of that Rās'is, which are therefore equal to $\frac{r \times \mathcal{T}}{30}$; from the given time substract the Bhogyāsus formulated above; then substract also the rising times of as many subsequent Rās'is as could be substracted. Let 'R' be the remainder of the time given. If t be the rising time in asus of the next Rās'i, $\frac{R \times 30^\circ}{t}$ gives the number of degrees by which the lagna has advanced in the next Rāsi. These degrees added to the previous Rās'is beginning from ν the equinoctial point give the Sāyana or the modern longitude of the lagna. From this Sāyana longitude if we substract the Ayanamsa, we have the Nirayana or the Hindu longitude measured from the Hindu Zero-point of the ecliptic. If, however, the given time after Sun-rise expressed in asus say 'a' falls short of the Bhogyāsus defined above, then, $\frac{a \times 30}{\mathcal{T}}$ where $\mathcal{T}$ is the rising time in asus of the Rās'i in which the Sun is situated, added to the longitude of the Sun, gives the Sāyana longitude of the lagna.

Comm. The substance of these verses, though appears to be simple, yet is complicated which can be better understood with the help of a figure (Ref. fig. 30).

Let SEN be the horizon, νER the celestial equator and νAL the ecliptic where L is the point of lagna. Required to find νL the Sāyana longitude of L from which if Ayanamsa be substracted we get the Hindu longitude. Let rA, AB, BC, CD, DF be the successive Sāyana Rās'is called Sāyana Meṣa, Sāyana Vriṣabha etc. (The Nirayana Rās'is also starting from the Hindu zero-point are called Nirayana Meṣa, Nirayana Vriṣabha etc. and if in Hindu Astronomy we use the words simply as Meṣa, Vriṣabha etc. especially in panchāngas ie. the Hindu almanacs, we have to construe them as belonging to the Nirayana or the Hindu system whose zero-point is called the first point of the constellation Aswini and not r).

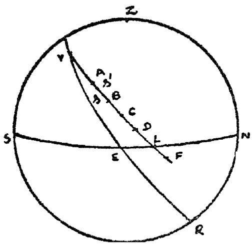


Fig. 30

rL = rD + DL = an integral number of Rās'is, say, 'n' of them ie. $n \times 30° +$ DL, where DL is the arc of the Rās'i carrying the Lagna L. The question then resolves itself into knowing how many Rās'is precede 'D' from r and what the measure of DL is. The data are (1) the Nirayana longitude of the Sun as computed by the methods of Hindu astronomy (2) The Ayanamsa of the year ie. 'ro' where o is the Hindu zero-point of the ecliptic. (3) The time after Sun-rise at the place, given in Sāvana units at which we are required to find the Lagna.

Finding the Lagna at a given point of time at a given place is not only necessary in astronomy but its importance is more in what is called horary astrology or Muhūrta S'āstra, whose purpose is to fix an auspicious moment for the performance of marriages etc. as well as in astrology in casting a horoscope.

The finding of the rising times of the various Rās'is at the equator, as well as at a given place was dealt with in the previous chapter. Those rising times are found in sidereal units, a sidereal day being divided into 60 nādis, or 60 × 60 Vinādis or 60 × 60 × 6 asus. These units are of constant magnitude since a sidereal day, which is

the period of diurnal rotation of the earth is of constant magnitude. In the data given above, the time is normally given in Sāvana units, ie. mean solar units. A Sāvana day is of $60 \times 60 \times 6 + 59$ asus, which is greater than a sidereal day by 59 asus because the mean Sun advance by $59' - 8''$ per day among stars and as such the Sun-rise the next day is belated by 59 asus approximately. The given time in Sāvana units could be converted into sidereal units as per the approximate ratio 21600: 21660 which means for every Sāvana nādi we have to add one asu or for every hour four seconds. Then the procedure of finding the Lagna will be a little different from what it would be if we proceed with the Sāvana units. The complexity mentioned before, arises out of the Sāvana units, and this has been explained by Bhāskara in Golādhyāya under the title तात्कालिकीकरणवासना in the beginning of the chapter called त्रिप्रश्नवासना. Now the procedure will be explained. Let (Fig. 30) S′ be the position of the Sun at the Sun-rise and S his position at the time at which the Lagna is to be found so thst the Sun has advanced by S′S ie. which measured in minutes of arc is called gatikalās. In a mean solar day these gatis-kalas would be 59, and in the given time after Sun-rise they will be proportional. The time given after Sun-rise pertains to the rising time of the arc S′L which we have in sidereal units. We shall first find the Lagna using sidereal units by measuring S′L, so that we may latter understand Bhāskara's reasoning for his stipulation of
on which basis he finds the Lagna with the Sāvana units taking the position of the Sun at S instead of S′.

We know the position of the sun at sun-rise ie. S′, so that the rising time of As′ ie. the previous arc in the Rāsi in which the Sun is situated is given by Bhuktasus defined above and the rising time of S′B the remaining arc of the Rāsi is given by Bhogyāsus. Subtract from the given time converted into sidereal units if they are not sidereal, the rising time of S′B ie. the Bhogyāsus, whose formula is $\frac{S'B \text{ (indegrees)} \times T}{30}$ where T is the rising of that Rāsi expressed in asus. Then subtract the rising times of as many subsequent Rās′is as could be subtracted ie. here from the figure the rising times of BC, CD. Then there remains the rising time pertaining to the arc DL from which we could calculate the magnitude of DL in degrees by the formula $\frac{R \times 30}{T}$ where 'R' is the remainder in time after subtracting the rising times of s′B, BC, CD and T the rising time of the Rās′i D.F. Then the longitude of L is $rA + AB + BC + CD + DL = 4 \times 30 + DL$ (In the figure shown rD cannot equal 4 Rās′is but will be far less than that but for illustration alone, we have represented it as consisting of 4 Rās′is).

The method of finding the longitude of L as above is quite plain being done on the basis of sidereal units and taking the position S′ of the Sun at Sun-rise. But Bhāskara adopts the position S and Sāvana units which compelled him to take pains to explain what is called तात्कालिकीकरण or obtaining the position S from the Sun-rise position S′. In fact the Sun is at S at the time at which the Lagna is to be found and we have to find DL as before. The argument advanced by Bhāskara is that the rising times of SB, BC, CD, DL measured in sidereal asus, will be just equal to the Sāvana asus, from Sun-rise, because the arcs S′L and SL differ by S′S ie. by the gatikalas pertaining to the time after Sun-rise. In other words. if the rising times of $S'A + AB + BC + CD + DL$ in sidereal asus give the sidereal time after Sun-rise, the rising times of $SA + AB + BC + CD + DL$ in the same sidereal asus give the Sāvana time after Sun-rise. Hence Bhāskara used the word तात्कालिकार्कोण in the beginning of the verse 2. Then he himself raised a purvapakṣa or the argument of an imaginary opponent namely "Is the time given measured after Sun-rise Sāvana (mean solar) or Nākṣatra (ie. sidereal)? If it be the former, how is it you are subtracting the rising times of *s*A, AB etc. which are sidereal from your Sāvana units? Further, should you not take the position of the Sun at Sun-rise namely S′, because the time given is what has elapsed after Sun-rise? Also, why should you complicate matters by accepting Sāvana units when the question is simple if dealt with sidereal units?

To this Bhāskara answers as follows –

"True it is, what you say. Generally in day to day life, time is given only in Sāvana units and not sidereal. Further you cannot avoid Sāvana measure, for, in the case of an arc moved by a planet, in its diurnal circle time is measured in the Sāvana units pertaining to the planet. (The Sāvana units of a planet are different from what they are for the Sun depending upon the arc moved by the planet in question during a day). These Sāvana units are what are termed Kshetra-Vibhāgātmika or what depend upon the arc covered in the diurnal path in contradistinction to the Kāla-Vibhāgatmika units or sidereal units. (In other words pure time is what is measured in sidereal units which is a standard measure, whereas time which has the bias of the motion of the planet also ie. which we seek to measure by the arc moved by a planet in its diurnal path, is Kṣetra-Vibhāgātmika). Thus having had to accept the Sāvana measure also, we seek to proceed on that basis, though we could convert the Sāvana measure into the sidereal and proceed without complication". The argument which Bhāskara gives for तात्कालिकीकरण is further as follows. The measure of the arc SL using the rising times in asus ie. sidereal units, is the Sāvana measure of the arc S′L done in sidereal units. Instead of subtracting the rising time of S′A from the given time converted into sidereal, we subtract the rising time of SA from the given Sāvana time and the result will be the same, for,

$-\text{SA} = -(\text{S}'\text{A} - \text{S}'\text{S}) = \text{S}'\text{S} - \text{S}'\text{A}$. which means subtracting SA from the time tantamounts to increasing by S′S and subtracting S′A. This increase by S′S is adding what have been defined as gati-kalas so that automatically the Sāvana units got converted into sidereal units, by taking the position S and subtracting SA instead of taking the position S′ and subtracting S′A. The result is the same. So, it is said 'तात्कालिकार्कस्य' in the beginning of the verse 2.

In the second case mentioned in verse 4, if the time given, falls short of the Bhogyāsus, then simply $\frac{x \times 30}{\text{T}}$ where x is the time given in asus and T the rising time of the Rāśi in which both the Sun and the lagna are then situated gives the arc in degrees which if added to the longitude of the Sun gives that of the lagna. Here also the position 'S' counts.

Verses 5 to 6½. To find conversely the time that has elapsed after Sun-rise given the lagna.

The Bhogyasus of the Sun and the Bhuktāsus of the Lagna together with the rising times of intermediate Rasis gives the time required.

If the Sun and the Lagna both be in the same Rāśi, then the arc in between them, multiplied by T and divide by 30, gives the time required.

If, however. the longitude of the Lagna falls short of that of the Sun, ie. if the Sun be below the horizon, (in this case SL $> 180°$) then finding the time of rising of SL and subtracting from a day, we have the time of the Lagna before Sun-rise.

However, here. there is one complication if we consider the तात्कालिकार्क ie. *s* the Sun at the given time. This position of *s* cannot be had unless we know the time, which is itself required. So we get in the first place the time pertaining to S′L, which is the time measured in sidereal units the position S′ being that of the Sun at Sun-rise. If we don't take recourse to convert this time in sidereal units to Sāvana units using the proportion between them, then the alternative is to obtain the position S using the time obtained and then calculate the time again in Sāvana units.

If the time given to find the Lagna, be sidereal, it goes without saying that we find it from S′. Also S′ being given and if the time after sun-rise is required in sidereal units for a given Lagna, the method of successive approximation is unnecessary.

Verse 7. To find the Lagna before Sun-rise called Vilōmalagna.

Suppose it be required to find the lagna before Sun-rise, given the time before Sun-rise. Obtain the then position of the Sun and find his Bhuktāsus; subtract them from the given time; from the remainder, subtract the rising times of as many Rās'is as could be, rās'is behind the Sun's position. If R be the remainder in the time after these subtractions, then R × 30/T where T is the rising time of is the next preceding Rās'i, together with $n \times 30$, where n the number of integral Rās'is subtracted and the arc of the next Rās'i by which the Sun has advanced in his Rāsi at the time of the Lagna (known from the position of S found) the sum total of these three items being subtracted from the position of S gives the longitude of L.

Comm. Easy.

Verse 8. To obtain the East-West line.

The East-West line is roughly the join of the extremities of the morning shadow as well as that in the after-noon of a guomon placed at the centre of a circle drawn on a plane with any arbitrary radius, when those shadows equal the radius of the circle. But this line is to be deflected keeping its western point ie. the extremity of the morning shadow fixed through a distance $\frac{K(\sin \delta_1 \sim \sin \delta_2)}{\cos \phi}$ at the eastern end perpendicular to it, where the above distance is measured in units, which measure K.

Comm. The east and west points are where the celestial equator cuts the horizon. The east point is thus the point where the Sun-rises when he is exactly at the vernal equinox. The question is how to draw the east-west line on a plane. For this we are asked to draw a circle with any radius on that plane. The plane is described here as अम्भःसुसमीकृतक्षिति that kind of surface as is determined by the surface of water there. Such a kind of surface forms approximately a horizontal plane not of course exactly because such a surface is really spherical, the earth being a sphere. But because the radius of the earth is sufficiently large, we can take such a surface to be a horizontal plane for all practical purposes. Having drawn a circle place the gnomon vertical at the centre. In the morning note the extremity of the shadow when it equals the radius. In the afternoon also mark the point when the shadow equals the radius. Join those two points. It represents rougbly the east-west line, roughly because the Sun's declination changes in between the two moments however small the change might be. Ignoring the change in the declination, this line will be east-west because of the following reason. The length of the shadow is 12 tan z where 12 units are the measure of the gnomon. But since the shadow on both the occasions is equal to the radius of the circle z, the zenith-distance will be the same on both the occasions. Then the spherical triangles PZS_1 and PZS_2 where P is the celestial pole, Z the Zenith and S_1 and S_2 are the positions of the Sun on the two occasion, are congruent their three sides being respectively equal, provided we take $PS_1 = PS_2$ ie. $90 - \delta_1 = 90 - \delta_2$ ie. $\delta_1 = \delta_2$ on both the occasions. When the two triangles are thus congruent, $P\hat{Z}S_1 = P\hat{Z}S_2$ ie. S_1 and S_2 are equidistant from the plane of the Prime-Vertical. Hence the extremities of the shadows will be equidistant from the East-West line; or this may be seen in another way; $S_1 S_2$ will be perpendicular to the meridiān plane and as such parallel to the plane of the Prime-Vertical.

The correction mentioned in the verse is known as the Agrāntara correction which was originally given by Chaturvedā chārya and then accepted by Srīpati. Why it is called Agrāntara is because it is a change in what are called Karnavrittāgras of the two occasions where we shall see in due course that the formula for Karnavrittāgra is $\frac{K \sin \delta}{\cos \varphi}$ *where K is hypotense of the gnomonic triangle formed* by the gnomon and its shadow S at any place and time. This correction is a very minute correction and as a matter of fact could be ignored. But the fact that the correction was cognized and correctly formulated testifies to the knowledge of the sphere which the above acharyas had. Assuming the formula of the Karnāgra here (it will be proved by us later in this chapter) if δ_1 and δ_2 be the declinations on the two occasions respectively the Karnāgras will be $\frac{K \sin \delta_1}{\cos \varphi}$ and $\frac{K \sin \delta_2}{\cos \varphi}$, K being equal on the two occasions because the shadows are equal. Hence the correction being the difference of the Agrās, it is $\frac{K (\sin \delta_1 \sim \sin \delta_2)}{\cos \varphi}$ as stated by Bhās-

We shall now prove it in modern terms from the spherical triangle PZS. We have the formula $\sin \delta = \sin \phi \cos z + \cos \phi \sin z \sin a$ where $PZS = 90 - a$, a being the Hindu azimuth measured from the East point. Multiply the above equation by K and divide throughout by $\cos \phi$ where K is called the Chāyākarṇa equal to $\sqrt{12^2 + S^2}$, S being the gnomonic shadow at the moment. Hence

$$\frac{K \sin \delta}{\cos \phi} = K \cos z \tan \phi + K \sin z \sin a \ (1)$$

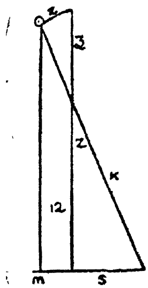


Fig. 31

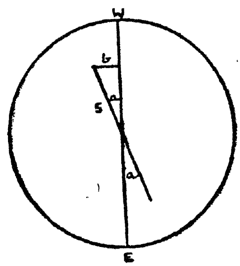


Fig. 32

But from Fig. 31, $K \cos z = 12$, $K \sin z = S$, (2) and from Fig. 32, $S \sin a = b$ where b is called the Chāyābhuja ie. Chāyābhuja $= K \sin z \sin a$ (3). Thus

But again from Fig. 33, when ⊙ the Sun at vernal equinox is on the meridian and as such has a meridian zenith-distance equal to ϕ, $12 \tan \phi = s$ where s is called the Vishuvat-chāyā or equinoctial shadow. Thus we have $\frac{K \sin \delta}{\cos \phi} = s + b$ (4).

Again if the Sun be on the horizon, from Fig. 34, E ⊙ is called the Agrā, A, so that from the triangle P

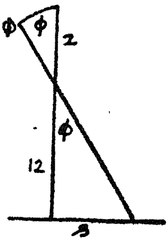


Fig. 33

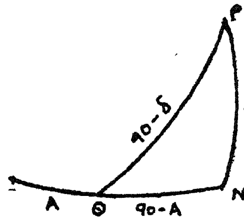


Fig. 34

$\cos(90 - \delta) = \cos\phi \cos(90 - A)$ ie. $\sin \quad = \sin A$ or

in the Hindu form H sin A $= R \dfrac{\sin}{\cos} =$ Agrajyā (5).

This Agrajyā is in a circle of radius R, and if it be reduced to a circle whose radius is K, is will be $\frac{K}{R} \times \frac{R \sin\delta}{\cos\phi} = \frac{K \sin\delta}{\cos\phi}$ *which is called Karnāgrā.* Hence we have *Karnāgrā* $= s + b$ which we shall write as $a = b + s$ (6). This is an important formula which is going to be formulated later in verses 72, 73. In the above formula, s being constant, by differentiating $\delta a = \delta b$ which means that the variation in the bhuja is on account of the variation in the Karnāgrā. If in Fig. 35, $C\omega'$, CE'' be morning and evening shadows when they are equal as per the verse under comment, $M\omega' = b$ the morning bhuja, $NE'' = b'$, the evening bhuja dE' is the variation in the bhuja ie. $b - b' = \delta b$ which is formulated and equal to δa. But $\delta a = \delta\left(\frac{K \sin\delta}{\cos\phi}\right) = \frac{K\,\delta(\sin\delta)}{\cos\phi}$, ϕ being constant and K also being constant because the shadow S is constant and $K = \sqrt{S^2 + 12^2}$ = constant on both the occasions. $\therefore\ \delta b = \delta K = \frac{K}{\cos\phi}(\sin\delta_1 - \sin\delta_2)$ as stated by Bhāskara.

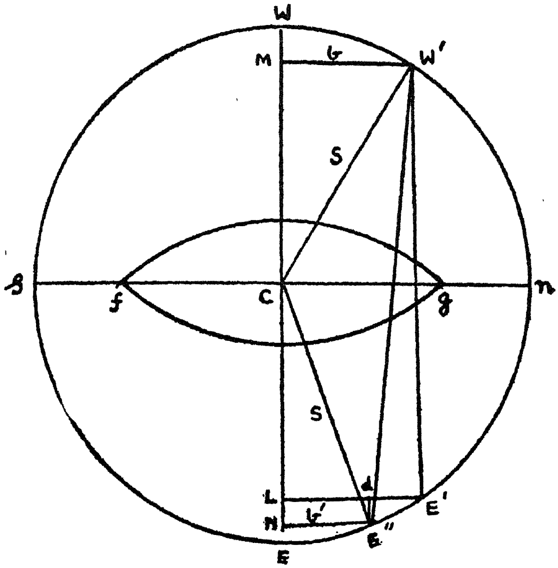


Fig. 35

9. Having determined the East-West line as mentioned, by drawing with a compass with the same radius on either sides of Eω (Eω is the East-West line through the centre of the circle drawn parallel to the rectified East-West line namely E′ ω′ where E″ ω′ is the approximate East-West line obtained by the join of the extremities of the shadows) two arcs which intersecting each other form a fish-like enclosure as shown in the figure and as such called Matsya meaning a fish, and joining the ends of the fish namely *f*, *g* and producing *fg*, we have the south and the north points *s* and *n*. Or again, the north-south line could be had from Fig. 36, where EC is a rod held in the direction of the north-pole as seen by the eye at *e* and EA and CB the lines indicated by the plumb-line called अवलम्बकसूत्र A and B being on the plane and AB joined passing through N, the north-point. Or again the directions could be determined as follows from a single shadow S namely Cω′ in Fig. 35,

calculating the bhuja $\omega' M$ and Koti CM; Bhuja and Koti being known, and the shadow $C\omega'$ being drawn, holding two rods whose lengths are equal to the bhuja and Koti perpendicular to each other, one extremity of the bhuja-rod being held at ω' and one extremity of the Koti-rod being held at C, and the rods making a right angle at M, then the bhuja-rod determines the north-south direction and the Koti-rod the East-West direction.

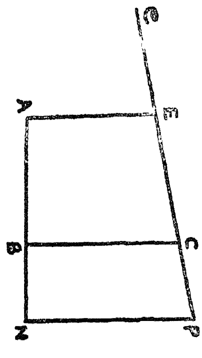


Fig. 36

Verse 10. The Bhuja is defined as the distance of the extremity of the shadow from the east-west line where the S'anku or gnomon is placed at the intersection of $E\omega$ and NS. Koti $= \sqrt{a^2 - b^2}$ $\therefore$ Koti will be in the East-West direction. So Chayā-koti = K sin z cos a (7).

Comm. Easy.

Verse 11. The Chayākarna, K is equal to $\sqrt{S^2 + 12^2}$, so that $\sqrt{K^2 - 12^2} = S$ or $\sqrt{(K + 12)(K - 12)} = S$.

Comm. Easy.

Verse 12. The S'anku is also called Nara or Nā. The zenith-distance of the Sun at Noon when the Sun is in *r* is the latitude of the place, called pala or Aksha; the altitude then is called lamba or colatitude.

Comm. The word S'anku we have previously used for the gnomon. It is also used for the H cosine of the zenith-distance and to differentiate it from the previous S'anku called Dwādasāṅgul'a-S'anku or twelve-unit-length S'anku, it is termed Mahā S'anku and occasionally Iṣta-S'anku. Mahā S'anku or *Iṣta S'anku* = *H cos z* (8). Thus in figure (37) $\odot M$ = H cos z. In the fig. where *gn* = gnomon, *go* = S the shadow, $\odot$ = the position of the Sun whose zenith-distance is $\odot z$. If z' be taken as the zenith $\odot z'$ measures the zenith-distance whereas if z be taken as the zenith $\odot z$ is the zenith-distance. The apparent inconsistency that both $\odot z'$ and $\odot z$ are taken as the zenith-distance is not there if we consider the zenith-distance as the angle $\widehat{\odot n z'} = \widehat{\odot o z}$. $O\odot = R$, $\odot z = z$, so that $\odot L = H \sin z$, and $LO = \odot M = H \cos z$ = S'anku or Nara or Nā. it is called Nara or Nā which means man, because a man may consider himself as a gnomon, which is called S'anku. So the word is also applied to the parallel H cos z parallel to the gnomon and called Mahās'anku. *H sin z is called Drigjyā* (9) zenith-distance is known as Nati because it is depression from the zenith. $\odot K$ (Fig. 37) is called the unnati or altitude.

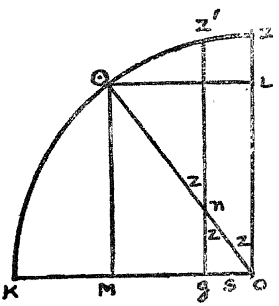


Fig. 37

Verses 13 *to* 17. Latitudinal triangles (Ref. Fig. 21).

(1) The right-angled triangle formed by the gnomon, the equinoctical shadow and the hypotenuse called Vishuvat-karna is the fundamental latitudinal triangle, which is like knowledge that will be the basis of all good things of the world, for example, respect, money, fame and happiness (Fig. 33).

(2) The second latitudinal triangle is that which is formed by H sin ϕ, H cos ϕ and the radius R of the sphere (Ref. $\triangle$ O Q L Fig. 38).

(3) The third latitudinal triangle is that formed by Kshitijyā, $S_1 B_1$ Krānrijyā $E_1 B_1$ and Agrajyā $E_1 S_1$ ie. the projected triangle $E_1 S_1 B_1$ of ESB (Fig. 21) on the

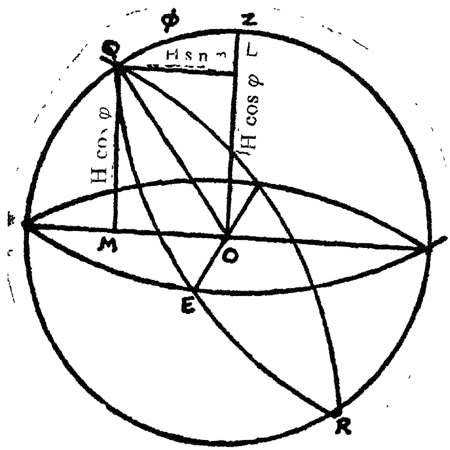


Fig. 38

plane of the meridian - where is the centre of the sphere.

(4) The fourth latitudinal triangle is $E_1 S_1 F_1$ the projection of ESF (Fig. 21) on the meridian plane where $E_1 F_1$ is the Sama-S'anku or the S'anku of the celestial body when it is on the prime vertical. $E_1 S_1$ is the Agrajyā as mentioned, $S_1 F_1$ is what is called Taddṛti.

(5) The fifth latitudinal triangle is $E_1 B_1 F_1$ where $E_1 B_1$ is Krāntijyā or H sin δ, $E_1 F_1$ is the Sama-S'anku defined above and $B_1 F_1$ is what is called the higher segment of the Taddṛti which is equal to Taddṛti minus Kujyā or Kshitijyā.

(6) The sixth latitudinal triangle is $E_1 D_1 B_1$ where $E_1 D_1$ is called the first segment of Agrajyā, $D_1 B_1$ is what is called un-mandala S'anku or H cos z of the celestial body when it is on the unmandala or the Equatorial horizon and $E_1 B_1$ Krāntijyā.

(7) The seventh latitudinal triangle is $D_1 S_1 B_1$ where $D_1 S_1$ is the second segment of the Agrajyā, $S_1 B_1$ is the Kujyā, and $D_1 B_1$ is the unmandala S'anku.

(8) The eighth latitudinal triangle is $B_1 L_1 F_1$ where $B_1 L_1$ is equal to the first segment of Agrajyā, $L_1 F_1$ is the higher segment of Agrajyā, $L_1 F_1$ is the higher segment of the Sama-Vritta-S'anku, and $F_1 B_1$ is the upper segment of Taddṛti mentioned before.

Comm. It was already mentioned that a latitudinal triangle is such a right-angled tri-angle constituted by the chords of the celestial sphere where the angles in the triangle are ϕ, $90 - \phi$, $90°$. The side opposite to φ is called the Bhuja, that opposite to $(90 - \phi)$ is called Koti and the third Karna. Such triangles are not only eight as have been mentioned, but many more will be there as mentioned by Bhāskara. They are all formed as mentioned by him by the intersections of the diurnal paths and the celestial equator with the circles of the sphere namely horizon, prime vertical, meridian, Equatorial horizon and declination circles. These circles clearly intersect at ϕ or $90 - \phi$. The eight triangles mentioned are those whose elements will be entering computations. There is another important latitudinal triangle with which we have to deal later namely that formed by what is called Hriti, S'anku, and S'ankutala (Fig. 39).

O_3K = Agrā ; O_3C = S'anku-tala ; CN = S'anku-bhuja = CK $\therefore$ Agrā = S'ankutala + S'anku-bhuja

$$\text{Agrā} = R\,\frac{H \sin \delta}{H \cos \phi}\,; \quad \text{S'ankutala} = H \cos Z \tan \phi$$

$$\text{S'anku-bhuja} = \frac{H \sin z\, H \sin a}{R} \text{ (Ref. fig. 40')}$$

$$\therefore \quad R\,\frac{H \sin \delta}{H \cos \phi} = H \cos z \tan \phi + \frac{H \sin z\, H \sin a}{R}$$

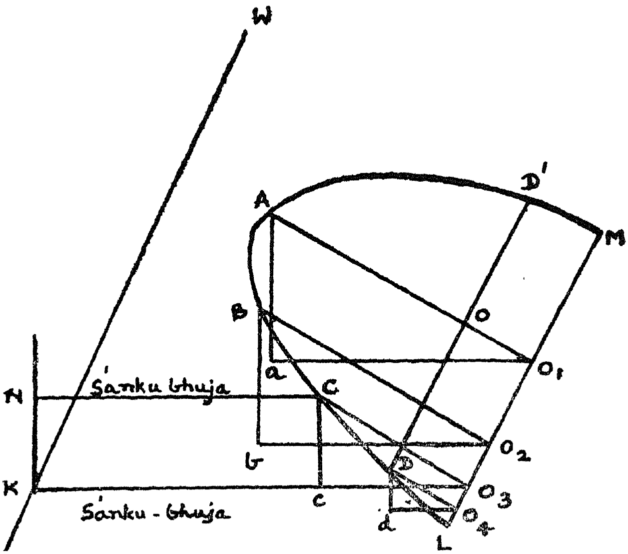


Fig. 39

or in modern terms $\sin \delta = \sin \phi \cos z + \cos \phi \sin z \sin a$ as derived from the triangle PZS. Formula I is with respect to the Mahā S'anku ⊙L of fig. 40; if it be reduced to the S'anku of the gnomon the S'anku-bhuja ⊙N becomes the Chāyābhuja pr which will be equal to

$$\frac{\text{H sin z H sin a}}{\text{R}} \times \frac{\text{K}}{\text{R}} = \text{K sin z sin a whereas,}$$

Agrā becomes $\dfrac{\text{R H sin}}{\text{H cos } \phi} \times \dfrac{\text{K}}{\text{R}} = \dfrac{\text{K sin}}{\text{cos}}$ which is called

Karnāgrā and S'anku-tala becomes

$$\frac{\text{H cos z tan } \phi \times \text{K}}{\text{R}} = \text{K cos} \qquad = 12 \tan \phi =$$

which is a constant quantity. It will be noted that Karnāgra differs from point to point since K differs from time to time during a day. In fact Karnāgra is the perpendicular dropped from the extremity of the shadow on the line parallel to the East-West line and north of it at a distance of the equinoctial shadow. As the shadow varies from time to time Karnāgra varies from time to time. Chayābhujā is the perpendicular dropped from the same extremity of the shadow on the East-West line. (Vide figure under verses 72, 73).

Let fig. 39 represent the diurnal path of a celestial body which is parallel to the Equator. Let A be the point where the celestial body culminates or crosses the meridian of the place, B the point where it crosses the prime vertical, C any arbitrary point of the orbit and D the point where it is on the Equatorial horizon, unmandala. Drop perpendiculars from A, B, C, D to the plane of the horizon, namely Aa, Bb, Cc, Dd. Let MO_1 O_2 O_3 O_4 L be the line of intersection of the plane of the diurnal circle with the horizon so that, LM is called the Udayāstasūtra, L being the point where the body rises and M where it sets. A line through D, the point where the diurnal path cuts the Equatorial horizon, drawn parallel Eω the East-West line will be a diameter of the diurnal circle and as such bisects the path. Draw perpendiculars from a, b, c, d to ML to meet it in O_1, O_2, O_3, O_4. By the theorem of three perpendiculars Ao_1, Bo_2, Co_3, Do_4 will be perpendiculars on ML in the plane of the diurnal circle. It is clear from the figure that all these right-angled triangles Ao_1a, Bo_2b, Co_3c, Do_4d are not only mutually similar but also are similar to the latitudinal triangles, in as much as the angles $\hat{A}$, $\hat{B}$, $\hat{C}$, $\hat{D}$, the angles between the vertical plane and the diurnal plane being equal to the angle between the planes of the prime Vertical and the Equator, are all ϕ and angles $\hat{a}$, $\hat{b}$, $\hat{c}$, $\hat{d}$ are right angles. Hence these triangles are also latitudinal. In fact Bbo_2, Ddo_4 were already included by us in the list of the eight latitudinal triangles since they are congunent to E_1 F_1 S_1 and D_1 B_1 S_1. As a matter of fact O_2B is Taddhṛti itself, O_4D Kujyā, Bb = Sama-S'anku, and Dd unmandala S'anku. O_2b is not actually the Agrajyā but parallel and equal to it, since Agrajyā is the H sine of SE of fig. 21, which is the perpendicular from S on $E\omega$. Similarly do_4 is not the second segment of Agrajyā but a parallel and equal segment. Thus Bb is the perpendicular distance between EE′ and FF′ ie. $E\omega$ and a parallel through F to $E\omega$ (fig. 21.) Dd is the perpendicular distance between DD′ and BB′; bo_2 is the perpendicular distance between $E\omega$ and SS′; do_4 the perpendicular distance between DD′ and SS′ (fig. 21); Bo_2 the perpendicular distance between FF′ and SS′; Do_4 the perpendicular distance between BB′, SS′. In this fig. 39, O_1A is called Hṛti, Co_3 = Ishta-Hṛti or any arbitrary hṛti. Taddhṛti Hṛti and Kujyā are special cases of Ishtahṛti. Hṛti is the maximum of Ishtahṛti. Calling Aa, Bb, Cc, Dd S'ankus in general ao_1, bo_2, co_3, do_4 are called S'ankutalas. Aa is called Dinārdha-S'anku or the S'anku of the mid-day; whereas Dd is the unmandala-S'anku and Bb Sama-S'anku. Cc is called Ishta-S'anku. Perpendiculars from A, B, C, D on the plane of the prime vertical are called S'anku-bhujas. Since B is on the prime-vertical itself, the S'anku-bhuja is zero and at this point BO_2 is Agrajyā. In the arbitrary case at C, the perpendicular from C on the plane of the prime-vertical being S'anku-bhuja, which is equal to the perpendicular from C on $E\omega$, and Co_3 being the S'anku-tala, and since the perpendicular distance between ML and $E\omega$ is the Agrajya, which is equal to the sum of O_3 C and the S'anku-bhuja S'anku-tala + S'anku-bhuja = Agrajya. (10) which is a different expression of (5). We shall present this analytically. Putting S'anku = H cos z, and using the latitudinality of Cco_3 $\frac{H \cos z}{12} = \frac{Co_3}{s} = \frac{Co_3}{K}$ applying similarity with the first fundamental latitudinal triangle where s = equinoctial shadow, and K the Viṣuvat-Karṇa. Hence we have

$$Co_3 = \text{Ishta-hṛti} = \frac{K}{12} H \cos z = \frac{H \cos z}{H \cos \varphi} = \frac{\cos z}{\cos \varphi} \quad (11)$$

$$Co_3 = \text{S'anku-tala} = \frac{s}{12} H \cos z = H \cos z \tan \varphi$$

$$\frac{H \cos z \; H \sin \varphi}{} = R \cos z \tan \varphi$$

From the triangle PZS, we have the formula
$\sin \delta = \sin \varphi \cos z + \cos \varphi \sin z \sin a$ written under verse 8

$\therefore \frac{\delta}{\cos \varphi} \cos z \tan \varphi + \sin z \sin a$ or in the Hindu form

$$R \sin \delta = H \cos z \tan \varphi + \frac{H \sin z \; H \sin a}{R}. \quad \text{I.}$$

We saw under verse 8 that $\frac{R \sin \delta}{\cos \varphi} = \frac{RH \sin}{H \cos \varphi}$

= Agrajyā (formula 5). Also we have from (12) above $H \cos z \tan \varphi$ = S'anku-tala. From fig. 40, if SM be drawn secondary to the prime-vertical, the right-angled spherical triangle SMZ gives

Sin $x = \sin z \sin a$ when SM $= x$ or in the Hindu form

$H \sin x = \frac{H \sin z \; H \sin a}{R}$; but $H \sin x$ we defined as

S'anku-bhuja, so that the equation I above may be written Agrajyā = S'anku-tala + S'anku-bhuja which is (10). *H sin a is called Dik-jyā* and *H sin z*, *Dṛk-jyā*. Formula (10) or (6) is the Hindu expression of the modern formula $\sin \delta = \sin \varphi \cos z + \cos \varphi \sin z \sin a$. *But the beauty lies in reducing* (10) to the formula *derived under verse* 8 *namely* $a = b + s$ ie. formula (6) to the *horizontal plane, introducing* the concepts of Karṇāgra *and Chāyābhuja* (Ref. fig. 40′). The lines corresponding to O_1A and O_3C in the Equatorial plane are called Antyā and Ishtāntya.

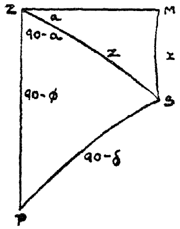


Fig. 40

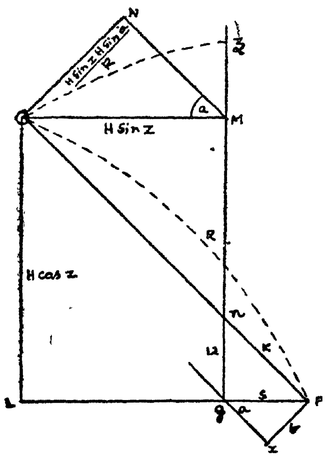


Fig. 40

Hṛti = $O_1A = OO_1 + OA$ = Kujyā + H cos δ or Dhujyā VI since $OO_1 = DO_4$ = Kujyā. The line in the Equatorial plane corresponding to Kujyā is *Charajyā which is equal to R tan φ tan* δ (13) so that we can write Kujyā = R sin δ tan φ = *H sin* δ *tan* φ (13′). Also from VI

Antyā = R + Charajyā = R + R tan φ tan δ (14) which gives us the duration of half the day where R gives 6 hrs and Charajya the increment in day due to φ and δ.

Verse 18. To obtain the magnitudes of the various chords or the elements of the latitudinal triangles.

The elements or the sides of all these latitudinal triangles are mutually derivable from similarity.

Comm. Easy.

Second half of Verse 18. The radius multiplied by the Kotis or Bhujas and divided by the Karnas gives H cos φ and H sin φ.

Comm. The seven triangles except the second, of the eight latitudinal triangles are compared here with the second. Remembering Bhujas are the sides of the triangles opposite to φ and Kotis opposite to 90 − φ,

$\frac{\text{Bhuja}}{\text{Karṇa}}$ in any triangle $= \frac{\text{Bhuja}}{\text{Karṇa}}$ in the second latitudinal triangle $= \frac{\text{H sin } \varphi}{\text{R}}$ $\therefore$ $\frac{\text{R} \times \text{Bhuja}}{\text{Karṇa}} = \text{H sin } \varphi$ Akshajyā or Palajyā (15) Similarly $\frac{\text{Koti}}{\text{Karna}}$ in any triangle is equal to the $\frac{\text{Koti}}{\text{Karṇa}}$ in the second latitudinal triangle $= \frac{\text{H cos } \varphi}{\text{R}}$ so that $\frac{\text{R} \times \text{Koti}}{\text{Karṇa}} = \text{H cos } \varphi =$ lambajyā (16).

e. The arcs of H sin φ and H cos φ are respectively the Akshams'as and lambams'as as they are called ie. latitude and colatitude. H sin φ and H cos φ are also obtainable thus $\sqrt{\text{R}^2 - \text{H sin}^2 \varphi} = \text{H cos } \varphi$ and $\text{H sin } \varphi =$ $\sqrt{\text{R}^2 - \text{H cos}^2 \varphi}$ Or again $\frac{\text{H sin } \varphi \times \text{Koti}}{\text{Bhuja}} = \cos \varphi$ and $\text{H cos } \varphi \times \frac{\text{Bhuja}}{\quad} = \text{H sin } \varphi$ where the Bhuja and Koti may belong to any latitudinal triangle.

Verse 20. Agrajyā can be had by multiplying Krāntijyā by Karṇa of any lat. triangle and divided by its Koti. Also Sama-S'anku $= \frac{\text{Karṇa}}{\text{Bhuja}} \times$ Krāntijyā and

$\frac{\text{Sama-S'anku} \times \text{Karna}}{\text{Koti}} =$ Taddhṛti.

Comm. The first of these statements pertains to the similarity of the third triangles to the others. The second of the statements pertains to the similarity of the fifth latitudinal triangle to others whereas the third pertains to that between the fourth and the others.

Thus Sama-S'anku or S.S.

$$\frac{H \sin \delta \times R}{H \sin \varphi} \text{ or } \frac{R \sin \delta}{\sin \varphi}$$

$$\text{Taddhṛti} = R\frac{H \sin \delta}{H \sin \varphi} \times \frac{R}{H \cos \varphi} = \frac{R \sin \delta}{\sin \varphi \cos \varphi} \quad (18).$$

Verse 21. $\text{Taddhṛti} = \frac{\text{Karṇa} \times \text{Agrajya}}{}$

Comm. This pertains to the similarity between the fourth latitudinal triangle and the others.

Latter half of Verse 21 *and first half of Verse* 22.

$$\text{Sama - S'anku} = \frac{\text{Taddhṛti} \times \text{Koti}}{\text{Karna}} = \frac{\text{Agrajyā} \times \text{Koti}}{\text{Bhuja}}$$

$$\frac{\text{Sama-S'anku} \times \text{Bhuja}}{}$$

Comm. The first statement is made out of the similarity between the fourth lat. triangle and the others whereas the second statement and the third as well are made out of the similarity between the third and others.

II half of Verse 22. Sama - S'anku = Uppor segment of Taddhṛti × Karna / Koti

$$\frac{\text{Uppor segment of Taddhṛti} \times \text{Karna}}{\text{Koti}}$$

Comm. The similarity is between the fifth triangle and others.

Verse 23. Kujyā = Krāntijyā × Bhuja/Koti

$$\text{Upper segment of Taddhṛti} = \frac{\text{Krāntijyā} \times \text{Koti}}{\text{Bhuja}} \text{ and}$$

Kujyā + Upper segment of Taddhṛti.

Comm. The first statement is through the similarity of the third triangle with others, whereas the second is through the similarity of the fifth with the others. The third statement is clear from Fig. 21.

Verse 24. $\frac{\text{Kujyā} \times \text{Bhuja}}{\text{Karṇa}}$ = second segment of Agrajyā $\frac{\text{Krāntijyā} \times \text{Koti}}{\text{Karṇa}}$ = first segment of Agrajyā and Agrā = Sum of the two segments.

Comm. The first statement is based on the similarity between the seventh triangle and others and the second on the similarity between the 6th and the others.

Verse 25. $\frac{\text{First segment of Agra} \times \text{Bhuja}}{\text{Koti}}$

= Un-mandala-S'anku and $\frac{\text{Krantijyā} \times \text{Bhuja}}{\text{Karṇa}}$

= Un-mandala-S'anku.

Comm. Both the statements are based upon the similarity of the sixth triangle and others. Thus un-mandala s'anku = U.S. = $\frac{\text{H} \sin \delta \times \text{H} \sin \varphi}{\text{R}}$ =

$$= \text{R} \sin \delta \sin \varphi.$$

Verse 26. $\frac{\text{First segment of Agrā} \times \text{Koti}}{\text{Bhūja}}$

= Un-mandala-S'anku = Kujyā × Koti/Karṇa.

Comm. The first statement is based on the similarity of the sixth latitudinal triangle and others and second that between the seventh and others.

Sama-s'anku – unmandala s'anku = upper segment of Sama-s'anku.

Verse 27. $\frac{\text{Agrā} \times \text{Bhuja}}{\text{Karṇa}}$ = Kujyā

Taddhṛti – Kujyā = upper segment of Taddhṛti.

Comm. The first statement is based on the similarity between the third triangle and the others and the second statement is evident.

Second half of verse 27. Other elements could be derived from what is already known and from what has been obtained. Also by alternando and invertendo we could pass from one element to the other and vice-versa.

Verse 28. $\text{Karṇa} = \sqrt{\text{Bhuja}^2 + \text{Koti}^2}$

$\text{Bhuja} = \sqrt{\text{Karṇa}^2 - \text{Koti}^2}$

$\text{Koti} = \sqrt{\text{Karṇa}^2 - \text{Bhuja}^2}$

Thus the third could be had from the other two in all the triangles.

Verse. There are sixty-three ways of obtaining H sin ϕ and H sin ϕ. On account of hundreds of ways of obtaining Agrajyā etc., there are an infinite number of ways of obtaining H cos ϕ etc.

Comm. Under verse 23 Bhāskara says that there are 98 ways of obtaining Taddhṛti. Taking the third latitude triangle, H sin δ conld be obtained in seven ways, from this H sin δ, Kujyā could be obtained in seven ways; hence, according to the principle of association namely that when one thing could be done in m ways and another in n ways, both the operations could be together performed in mn ways, so Kujyā could be obtained in $7 \times 7 = 49$ ways; similarly the upper segment of Taddhṛti could be had in 49 ways; so that adding the two Taddhṛti could be obtained in 98 ways.

Similarly suppose we have to find H sin δ. H cos ϕ could be found in seven ways and from H cos ϕ, H sin δ could be found in seven ways. Thus H sin ϕ could be found in 7×7 ways $= 49$ ways. From R, H sin φ could be found in seven ways by using similarity with the other seven latitudinal triangles except the second. Also obtaining H cos φ in seven ways and using the formula H sin $\varphi = \sqrt{R^2 - H\cos^2\varphi}$ we have seven more ways. Thus in all there are $7 \times 7 + 7 + 7 = 63$ ways. Similarly H cos φ could be found in 63 ways. Extending this to Agrajyā etc. which could be in as many or more ways themselves finding H sin φ therefrom means again finding it in 69×69 ways and so on. Since there is no end in counting all these ways, it is said that there are infinite ways to find it. The word 'infinite' here connotes only a very large number of ways not exactly what we mean by the word 'infinity'.

Verse 30. To find what is known as Koṇa-S'anku.

As a first approximation take Koṇa-S'anku $= \sqrt{R^2 - 2A^2}$ where A = Agrajyā and Koṇa-S'anku means H cos z when the azimuth is equal to 45°.

Then take Agrajyā $\pm$ the above Koṇa-S'anku $\times \frac{s}{12}$ = S'anku-bhuja $= b$ (say) then again Koṇa-S'anku $= \sqrt{R^2 - 2b^2}$.

Then again take Agrajyā $\pm$ the above Koṇa-S'anku $\times \frac{s}{12}$ as the new bhuja and proceeding thus by the method of successive approximations, we arrive at a constant value which gives the Koṇa-S'anku.

Comm. Bhāskara gives later the method of obtaining H cos z ie. the S'anku pertaining to any zenith-distance. So, he need not have given a separate treatment for this Koṇa-S'anku. But in as much as Brahmagupta and other previous writers gave it he has also given the same. He gives here the method of finding the Koṇa-S'anku by the method of successive approximations as given by S'ripati.

We saw before that Agrā = S'anku-tala + bhuja. So, in the first place as a first approximation, Agrā is taken as S'anku-bhuja. Since, when $z = 45°$ the perpendiculars from the celestial body on the planes of the prime-vertical as well as meridian are equal, and since the perpendicular on the plane of the prime-vertical is called S'anku-bhuja = b (say) $2b^2 = H \sin^2 z$. This is so because $H \sin^2 z$ = Sum of the squares of the perpendiculars on the planes of the meridian and prime-vertical. But $H \sin^2 z = R^2 - H \cos^2 z$. $\therefore R^2 - 2b'^2 = H \cos^2 z$. So, taking Agrā as the bhuja b as a first approximation, $\sqrt{R^2 - 2b^2}$ gives us $H \cos z$. From this using formula III under latitudinal triangles namely $H \cos z \times \frac{s}{12}$ = S'anku-tala, obtain the approximate S'anku-tala, from the approximate $H \cos z$ got above. Now using the formula Agrā = S'anku-tala + S'anku-bhuja obtain S'anku-bhuja as Agrā $\pm$ S'anku-tala, where the +ve sign is taken when the Sun has a southern declination, and the difference sign when the declination is north. Taking this S'anku-bhuja, b', Koṇa-S'anku is now $\sqrt{R^2 - 2b'^2}$. In the first place we took the Agrā itself as the bhuja; but here we have a better approximation for the S'anku-bhuja. From this Koṇa-S'anku again, obtain a still better approximation for S'anku-bhuja and proceeding thus till a constant value is obtained, we have the required Koṇa-S'anku. *This is a beautiful example where the method* of successive approximation was used by the Hindu Astronomers to a good advantage. It will be noted here that the S'anku-tala is always treated as extending south ie. the S'anku-tala will be south of the S'anku since India's latitudes are all north. Also it is said that when the Sun has southern declination, A + S = B and when northern A − S = B when A = S'anku-Agrā or simply Agrā (in contradistinction to Karnāgrā reduced to a circle of radius K the chayākarṇa) S = S'anku-tala and B = S'anku-bhuja. This convention of signs is to be correlated with the modern. We have from the formula derived out of PZS, $A = S + B$. According to modern convention when δ is north, it will be taken to be positive and when a is to the north of the East point it also will be taken to be positive so that, (1) when δ is north and a north, $A = S + B$ ie. $B = A - S$; this accords with the Hindu convention namely 'सौम्येत्वन्तरम्' (2) when δ is south and a south, $-A = S - B$ so that $B = A + S$; this also accords with the Hindu convention, namely याम्ये योगः (3) But, however, when δ is north and a south ie. when the Sun having northern declination comes to the south of the prime-Vertical, $A = S - B$ so that $B = S - A$. This accords with the Hindu convention if only we take $B = | A - S |$ when δ is north.

Bhāskara makes two statements at the end of the commentary under this verse namely that when δ is south and $A > 2431$, there will be no Koṇa-S'anku and that when δ is north and $s > 17'' - 5'''$ there will be four Koṇa-S'ankus. We have to verify these statements. The first statement is evident because H sin (Agrā) > H sin 45° ie. > 2431′, no Koṇa-S'anku will be formed above the horizon because the diurnal path above the horizon will be to the south of the points K, K′ on the horizon where $EK = 45°$ and $\omega K' = 45°$ where E, ω are the east and west points and K and K′ respectively lie on the eastern and western horizons between E and S and ω and S, S being the south point. Regarding the second statement, in order that there may be a Koṇa-S'anku in the north Agrajyā > H sin 45 ie. $\frac{\sin \delta}{\cos \varphi} > \sin 45°$ ie. $\sin \delta > \sin 45° \cos \varphi$. Taking the max. value for δ namely 24°,
$\log \sin 24° > \log \sin 45 + \log \cos \varphi$ ie. $9.6093 > 9.8495 + \log \cos \varphi$ ie. $\log \cos 45 < 9.7598$ $\therefore$ $\varphi > 54° - 54'$ $\therefore$ $\tan \varphi > 1.4229$ $\therefore$ $s = 12 \tan \varphi > 17'' - 4.5'''$ which is taken by Bhaskara as $17'' - 5'''$. For a lesser value of δ, a still greater value of φ will be required

as may be seen by taking $\delta = 20°$, $\varphi > 61° - 6'$. Thus Bhāskara gave the minimum latitude which could enjoy four Koṇa-S'ankus.

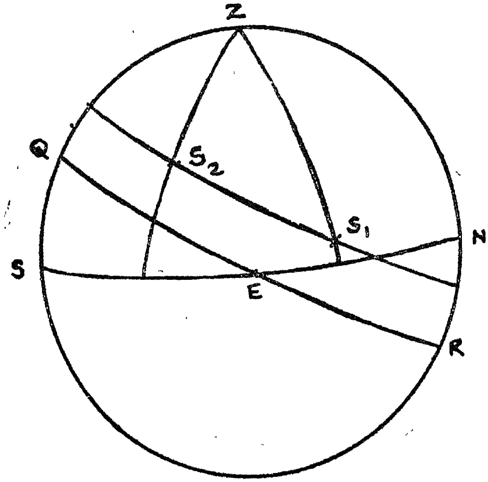


Fig. 41

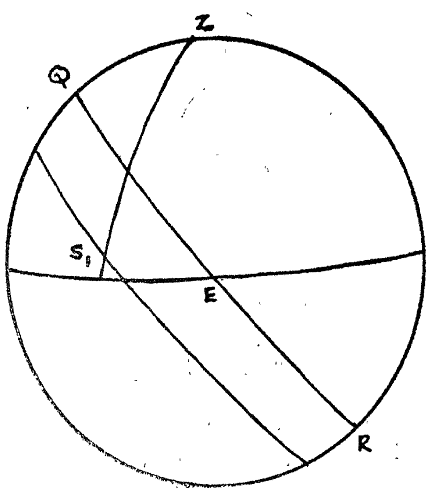


Fig. 42

From fig. 41, it is clear that there are two Koṇa-S'ankus at S_1 and S_2 during the forenoon and similarly two at S_1' and S_2' in the afternoon where S_1' and S_2' are the symmetrical points of S_1 and S_2. From fig. 42, it is clear that if Agrā $<$ H sin 45° when δ is south, there will be one Koṇa-S'anku in the forenoon at S_1 and one in the afternoon at the symmetrical S_1'.

Verses 31 *and* 32. H cos z at noon known as Dinārdha-S'anku.

By 'northern hemisphere' it is meant that the Sun is in the northern hemisphere ie. his S'āyana longitude ie. modern longitude lies between 0° and 180°, and 'the southern hemisphere' means that the Sun's longitude lies between 180° and 360°. The direction of δ may be got from the above convention. The latitude and colatitude are always deemed as south and north respectively.

The latitude and colatitude being 'added to subtracted from or being decreased by' as the case may be, the declination, we have the zenith-distance and the altitude of the celestial body at Noon. The zenith-distance and the altitude are mutually complements.

Comm. In Hindu Astronomy the words "उत्तरगोले" "दक्षिणगोले" are very often used to connote that the Sun is on the north or the south of the celestial equator respectively, so that the declination could be automatically known to be north or south respectively. Regarding the latitude, the peculiarity in Hindu Astronomy is that what we call north latitude in modern astronomy *is construed as south* in as much as the celestial equator gets depressed south in northern latitudes. The colatitude SQ in fig. 41 on the other hand extends north from the south, so that, it is construed as north.

The word 'Samskāra' is used in Hindu Astronomy in the meaning given above in the translation. For example in the equation $A = S + B$, we say that the Bhuja is had by a Samskāra between A and S. The meaning of Samskāra given by Bhāskara is "समदिशोर्योगः भिन्नदिशोरन्तरम संस्कारः" Latitude being regarded as southern, if the Sun's declination is 12° north and the latitude 20°, then as they are of opposite direction, effecting the Samskāra as directed $20 - 12 = 8$ = zenith-distance (South) = Nata as it is called similarly $70 + 12 = 82$ = Altitude = Unnata; here we have added because, both lamba and declination are north. Similarly when $\delta = 24°$ north, and $\varphi = 20°$ as before (south)

$24 - 20 = 4°$ = zenith-distance (north) = Nata

$70 + 24 = 94$ = unnata (north). But, we take $180 - 94 = 86°$.

In the above working in the first case we found $\varphi - \delta$, whereas in the second we found $\delta - \varphi$. This difference in treatment is not taken objection to, since, the word Antara is used to take the positive value of the difference alone and so in the first instance the nata is pronounced as south, whereas in the second it is pronounced north.

In modern astronomy, however, we have the formula $z + \delta = \varphi$, considering z as positive if south, δ and φ positive if north. Here $8° + 12° = 20°$ (first case cited above) and $(-4°) + 24° = 20°$ (2nd case, z being negative, for, it is north. In the Hindu symbolism we have to pronounce separately when z is south or north, whereas in modern symbolism the sign alone informs its direction. Similarly in the equation $A = S + B$, we have to pronounce 'north bhuja' or 'south bhuja' as the case may be, whereas having a convention that δ is +ve when north. and also the Hindu azimuth (measured from the East point) the sign of bhuja indicates its direction. In other words we differentiate the two cases $A - S$ and $S - A$ giving them signs and deducing the direction of the bhuja from the sign itself without an appeal to a picture or without ascertaining whether the northern Agrā prevails over the Southern S'anku-tala or the Southern S'anku-tala prevails over the northern Agrā. Thus the Dinārdha S'anku in symbolism = $H \cos(\varphi \pm d)$ (20).

Verse 33. Here at noon, Dṛg-jyā is the H sine of nata and the S'anku is H sine of unnata.

Second half of 33 and first half of Verse 34.

The product of R and the unmandala-S'anku divided by Charajyā is called Yaṣti. The Yaṣti increased by Un-mandala-S'anku gives H cos z according as the Sun is north or south of the equator.

Comm. Unmandala-S'anku is H cos z when the Sun is on the unmandala. From the sixth latitudinal triangle, wherein Unmandala-S'anku is Bhuja and Krāntijyā Karṇa, so by comparing with the second latitudinal triangle (or rather operating with the second triangle to signify the Hindu method).

$$\frac{\text{Krāntijyā} \times \text{Bhuja}}{\text{Karṇa}} = \text{Unmandala-S'anku}$$

$$= \frac{H \sin \delta \times H \sin \varphi}{R} \text{ (already}$$

We saw before Charajyā = $R \tan \varphi \tan \delta$. Hence as directed in the verse $\frac{R \times H \sin \varphi \, H \sin \delta}{R \times R \tan \varphi \tan \delta} = \text{Yaṣti}$

$$= \frac{H \cos \varphi \, H \cos \delta}{R} \text{ (21).}$$

$$\therefore \; H \cos z \text{ (at Noon)} = \frac{H \cos \varphi \, H \cos \delta}{R} \pm$$

$\frac{H \sin \varphi \, H \sin \delta}{R}$ according as the Sun is on the north or south of the equator.

Hence H cos z (at Noon) = Natajyā
$\frac{\text{H} \cos \varphi \ \text{H} \cos \delta \pm \text{H} \sin \varphi \ \text{H} \sin \delta}{\text{R}}$ or in modern symbolism cos ($\varphi \mp \delta$) already derived under (20).

We shall now show how the formulation is done by the simple rule of three (Ref. fig. 39). If a parallel through $o_4 d$ is drawn to cut Aa at a_1, then Aa_1 is called the Yaṣṭi, which is vertical. The triangles Aoa_1 and Do_4d are similar so that $\frac{\text{A}a_1}{\text{A}o} = \frac{\text{D}d}{\text{D}o_4}$ $\therefore$ $\text{A}a_1 = \frac{\text{DD} \times \text{A}o}{\text{D}o_4}$.

But $\frac{\text{A}o}{\text{D}o_4} = \frac{\text{R}}{\text{Charajyā}}$ for, all the lines of the diurnal circle and the equator stand in the ratio

$$\frac{\text{H} \cos \delta}{\text{R}} = \frac{\text{A}o}{\text{R}} = \frac{\text{D}o_4}{\text{Charajyā}} = \frac{\text{Kujyā}}{\text{Charajyā}} \therefore \frac{\text{A}o}{\text{D}o_4} = \frac{\text{R}}{\text{Charajyā}}$$

$\therefore$ $\text{A}a_1 = \frac{\text{Unmandala S'anku} \times \text{R}}{\text{Charajyā}}$ = Yaṣṭi as formulated.

Now Dinardha S'anku = Aa = Aa_1 + $a_1 a$ = Aa_1 + Dd = Yaṣṭi + Unmandala S'anku. It is evident from fig. 21 why in the northern sphere the sum is to be taken whereas in the southern, the difference is to be taken.

Latter half of verse 34. Definition of Hṛti and Antyā.

The sum or difference of Dyujyā and Kujya will be similarly Hṛti, whereas the sum or difference of Charajyā and radius will be Antyā.

Comm. We defined formerly Hṛti and Antyā our commentary on the latitudinal triangles. From fig. 8 Hṛti = o_1A = $o_1 o$ + oA = o_4D + oA = Kujya + H cos (Dyujyā) (22).

In the parallel great circle, the Equator we have therefore Antyā = Charajyā + R (already derived).

Verse 35. $\text{Antyā} = \dfrac{\text{Hṛti} \times R}{H \cos \delta} = \dfrac{\text{Hṛti} \times \text{Cha}}{\text{Kujyā}}$

$\therefore \text{Hṛti} = \dfrac{\text{Antyā} \times H \cos \delta}{R} = \dfrac{\text{Antyā} \times \text{Kujyā}}{\text{Charajyā}}$ by what is called Guna-ccheda-Viparyaya ie. alternando.

Comm. Evident.

Verse 36. To obtain Dinārdha-S'anku from Antyā and Hṛti

$\dfrac{\text{Antyā} \times \text{un-mandala S'anku}}{\text{Charajyā}} = \dfrac{\text{Hṛti} \times 12}{K} =$ S'anku.

Comm. The second formula is derived from the similarity of $\triangle$ Ao_1a (fig.) with the first latitudinal triangle. From the similarity of Aao_4, Ddo_4 fig. 39,

$Ao./Do_4 = \dfrac{Aa}{Dd} \quad \therefore \; Aa = \dfrac{\text{Hṛti} \times \text{unmandala-S'anku}}{\text{Kujyā}}$

But Aa = Dinārdha-S'anku

$\therefore$ Dinārdha S'anku $= \dfrac{\text{Hṛti} \times \text{U.S.}}{\text{Kujyā}}$ (U. S. = Un-mandala-S'anku.)

But Kujyā Charajyā

D.S. (Dinārdha-S'anku) $= \dfrac{\text{Charajyā}}{\text{Antyā}} \times$ U.S.

i × 12

Verse 37. Meridian Zenith distance.

$\text{Agrā} \pm \dfrac{\text{Hṛti} \times \text{bhuja of a lat. triangle}}{\text{Karṇa of a lat. triangle}} =$

where z is the meridian zenith distance.

Comm. This formula is a special case of the formula A = S + B since the H sine of the meridian zenith distance is the S'anku-Bhuja at noon. From fig. 39

$$\frac{\text{Hṛti} \times \text{bhuja of a latitudinal triangle}}{\text{Karṇa of a latitudinal triangle}} = O_1a = \text{Dinārdha}$$

S'ankutala.

The operation of sign has been already explained.

Verse 38. An alternative method.

The meridian zenith distance of the Sun can be had also by the formula (Hṛti $\pm$ Taddhṛti) $\frac{\text{B.L.T.}}{\text{K.L.T.}}$ where B.L.T. and K.L.T. are the bhuja and karṇa of any latitudinal triangle.

Comm. (Ref. figures 43 and 21). Let E_1 be the centre of the armillary sphere so that QE_1R is the diameter of the celestial equator which is on the median plane. Let $S_1 F_1 S$ be the diameter of the diurnal circle of the Sun, which is also on the meridian plane so that $S_1 F_1$ is the Taddhṛti, $S_1 S$ is the Hṛti and S the position of the Sun on the meridian.

H sin z = SM = SF_1 sin $\widehat{F_1}$ = SF_1 sin ϕ = (Hṛti − Taddhṛti) $\times \frac{s}{K}$ where s/K can be replaced by $\frac{\text{B.L.T.}}{\text{K.L.T.}}$ (s=equinoctical shadow and k the Viṣuvat-Karṇa).

In the Southern sphere, H sin z = $S'N$ = $S'F_1'$ sin ϕ = $(S'S_2 + S_2F_1')$ sin ϕ = (Hṛti + Taddhṛti) $\times \frac{s}{k}$.

Verse 39. Still another way of obtaining the *m. z. d.* (meridian-zenith-distance).

R − H versin (altitude) = H sin z

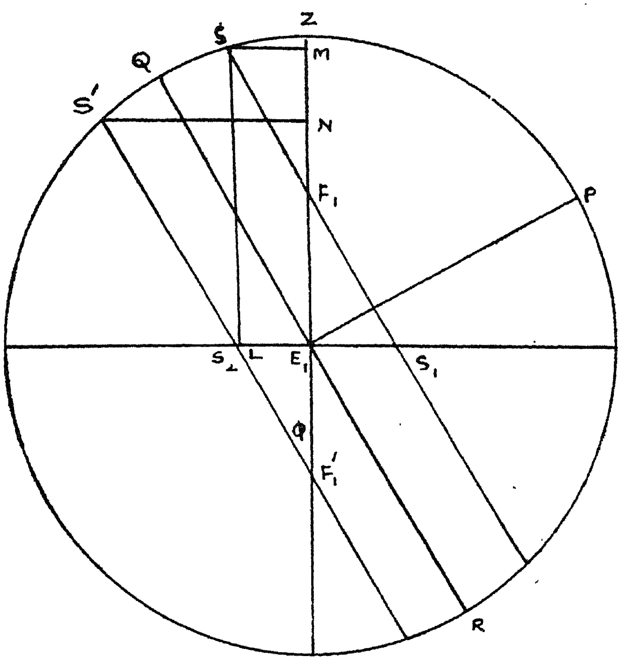


Fig. 43

This formula gives not only the meridian zenith distance but H sine of the zenith-distances of the Sun when he is on the Koṇa-Vṛtta or prime-vertical or unmandala.

Comm. From fig. 43, H sin z = SM = *s*L = R − H versine (S*s*) = R − H versine (altitude) as given. Since R − H versine (altitude) = R − {R − H cos (90 − z)} = R − (R − R sin z) = H sin z, so this formula applies wherever the Sun be. This is almost begging the question as H sine of z is being sought through H versine of (90 − z).

First half of the verse 40. To obtain the shadow S and K the Chayakarṇa of any shadow

$$\frac{\text{H sin z} \times 12}{\text{H cos z}} = \text{S and } \frac{\text{R} \times 12}{\text{H cos z}} = \text{K.}$$

Comm. (Ref. fig. 44).

$$\frac{12.\ \text{H sin z}}{\text{H cos z}} = 12 \tan z = \text{S.} \qquad \text{Also } \frac{12}{\text{K}} = \cos z = \frac{\text{H cos z}}{\text{R}}$$

so that $\frac{12\ \text{R}}{\text{H cos z}} = \text{K.}$

The Hindu method of looking at this through the similarity of $\triangle$S OM$\odot$ and Ogn the gnomonic triangle. is as fellows. $\odot$M is called Mahā-S'anku ie. H cos z ; $\odot$L is Dṛkjya or

H sin z = OM. $\frac{12}{\text{H cos z}} = \frac{\text{S}}{\text{H sin z}}$ so that S =

12 H sin z/H cos z. Also,

$$\overline{O\odot} - \frac{12}{\text{H cos z}} \text{ ie. } \frac{\text{K}}{\text{R}} = \frac{12}{\text{H cos z}} \quad \therefore \quad \text{K} = \frac{12\ \text{R}}{\text{H cos z}}.$$

It will be noted that fig. 44 pertains to any vertical plane.

Second half of Verse 40. The Dinārdha-Karṇa is equal to $\frac{\text{R} \times k}{\text{Hṛti}}$ where k is the Viṣuvat-Karṇa.

Comm. The formula is derived through twice applying the rule of three or what is the same, through the similarities of two sets of triangles

From fig. 39, $\frac{O_1A}{Aa} = \frac{\text{Hṛti}}{\text{Dinārdha-S'anku}} = \frac{k}{12}$ (a)

where K is the required Chayākarṇa.

Dividing (a) by (b)

$\overline{12} - 12 \wedge \text{K} \ .. \quad - \overline{\text{Hṛti}}$

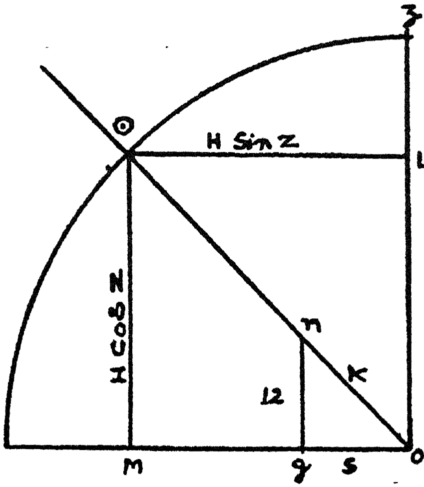


Fig. 44

Verse 41. Alternate method of obtaining K

101530/H sin λ = para (say) where λ is the Sāyana longitude of the Sun ; then, $\frac{\text{Para} \times k}{s} = K$ where K is un-maṇḍala-Karṇa

First half of Verse 42. To obtain K when the Sun is on the prime-vertical— Para $\times$ s/k = Samavṛttakarṇa.

Comm. From fig. 19, from the similarity of triangles

CA

$$H \sin \delta \quad \frac{H \sin \lambda \ H \sin \omega}{R} \quad (a)$$

Then consider the similarity of the first and the sixth latitudinal triangles ; then $\frac{\text{Unmandala S'anku}}{\text{Krāntijyā}} = \frac{s}{k}$ (b)

where s is the equinoctial shadow and k the Viṣuvat-Karṇa. Again taking that ⊙ the Sun lies on the unmandala in figure 44, $\frac{\text{H cos z}}{12} = \frac{\text{R}}{\text{K}} = \frac{\text{Unmandala S'anku}}{12}$ (c)

Eliminating Krāntijyā and Unmandala S'anku from (a), (b) and (c) $\frac{\text{Unmandala S'anku}}{\text{H sin } \lambda \text{ H nis } \omega/\text{R}} = \frac{s}{k}$

$12\text{R} / \text{H sin } \lambda \frac{\text{H sin } \omega}{} = \frac{}{}$ ie. $\frac{}{\text{KH sin } \lambda \text{ H sin } \omega}$

$\frac{12\text{R}^2 \times k}{\quad \text{n } \omega}$. Here $\frac{12\text{R}^2}{\text{H sin } \omega} = \frac{12 \times 3438^2}{1397}$ $= 101531$; but Bhāskara has taken 101530 taking a more correct value of R. Then $\frac{101530}{\text{H sin } \lambda}$ is symbolized as para so that para $\times \frac{k}{s}$ =K=Unmandala Karṇa. Regarding the Samavṛttakarṇa, in the place of (b) above we have $\frac{\text{Sama-S'anku}}{\text{Krāntijyā}} = \frac{k}{s}$ (b′) by the similarity between the first and the fifth latitudinal triangles. Equation (c) holds good with respect to any H cos z and the corresponding K since 12 R = K × S'anku and 12 R is a constant. Noting therefore $\frac{\text{R}}{\text{K}'} = \frac{\text{Sama-S'anku}}{12}$ (c′) eliminating Krāntijyā and Sama-S'ankn among (a), (b′), (c′), we shall have

$$\text{K} = \frac{12\ \text{R}^2\ s}{k\ \text{H sin } \lambda \text{ H sin } \omega} = \text{para} \times \frac{s}{k} \text{ as stated.}$$

Second half of Verse 42. To obtain the Dinārdhakarṇa from the Unmandalakarṇa.

$$\frac{}{\text{Antyā}} \quad \frac{\text{Charajyā}}{} = \text{Dinārdhakarṇa.}$$

Comm. We have equation (c) above stating 12 R = K × S'anku. (c) But

$$\frac{\text{Iṣta S'anku}}{\text{Iṣta Hriti}} = \cos\varphi = \text{constant} = \frac{\text{Dinārdha S'anku}}{\text{Hṛti}}$$

$$= \frac{\text{Sama-S'anku}}{\text{Taddhṛti}} = \frac{\text{Unmandala S'anku}}{\text{Kujyā}}$$

Again by virtue of the proportionality of

$$\frac{\text{Iṣta Hṛti}}{\text{Iṣtāntya}} = \frac{\text{Hṛti}}{\text{Antyā}} = \frac{\text{Kujyā}}{\text{Charajyā}} \quad (e)$$

We nave $\dfrac{\text{Ishta-S'anku}}{\text{Ishtāntyā}} = \dfrac{\text{Dinārdha S'anku}}{\text{Antyā}}$

$$= \frac{\text{S'anku}}{\text{Charajyā}}$$

$\therefore$ Iṣta Karṇa $\times$ Iṣtāntya $=$ Dinārdha Karṇa $\times$ Antyā
$=$ Unmandala Karṇa $\times$ Charajyā (f) (25)

$$\therefore \text{ Dinārdha Karṇa} = \frac{\text{Unmandala Karṇa} \times \text{Charajyā}}{\text{Antyā}}$$

as stated in the verse.

Verse 43. $\dfrac{\text{Unmandala Karṇa} \times \text{Kṣitijyā}}{\text{Hṛti}}$

$$= \frac{\text{Sama Vṛtta Karṇa} \times \text{Tε}}{\text{Hṛti}}$$

Comm. From (c) and (d) above Dinārdha Karṇa $\times$ Hṛti $=$ Sama Karṇa $\times$ Taddhṛti $=$ Unmandala Karṇa $\times$ Kujyā (g). Khitijyā is the same as Kujyā.

From this the statement follows :

Verse 44. The ancient Achāryas found the gnomonic shadows when the Sun is on the meridian, prime-vertical and the Kona-Vṛtta (ie, Vertical when the northern or southern Hindu azimuths are 45°) by different methods. I consider him to be the very Sun illuminating the lotus-faces of aitronomers, if anybody could give a method to find the shadow in any required direction, which holds good in all cases universally.

Comm. Evident.

Verse 45. Definition of Dikjyā H sin (azimuth). The angle between any vertical and the Prime-Vertical measured on the horizon is what is called Digamsa and its H sine is known as Dik-jyā either in the Eastern hemisphere or the Western.

Comm. In modern astronomy azimuth is measured along the horizon from the north point towards the east point round the horizon. In Hindu Astronomy however, the azimuth is measured from the East point on either side and from the West point also on either side specifying whether it is north or south.

Verse 46 *and first half of* 47. To obtain the gnomonic shadow in any arbitrary direction.

Assume $\frac{Rs}{\text{H sin } a}$ as the equinoctial shadow and obtain the H sine of the corresponding latitude L. Then the product of that H sin L and H sin δ divided by H sin φ will give H sin D where D is a hypothetical declination. With the new L and this D, as the hypothetical latitude and declination, obtain the meridian zenith distance by the formula Z + D = ϕ, and through this *m.z d.* obtain the shadow, which will be the shadow in the required direction namely 12 tan ($\phi \overset{+}{\sim}$ D).

Comm. Let gL be the gnomonic shadow on the equinoctial day in a given direction given by a° Digamsa (the Hindu azimuth) and let gN be the shadow in the same direction on any day. (fig. 45) We know that the extremity of the gnomonic shadow on the equinoctial day traces a straight line parallel to the East-West line Eω at a distance of the equinoctial shadow *s* because the Equatorial plane passing through the foot of the gnomon

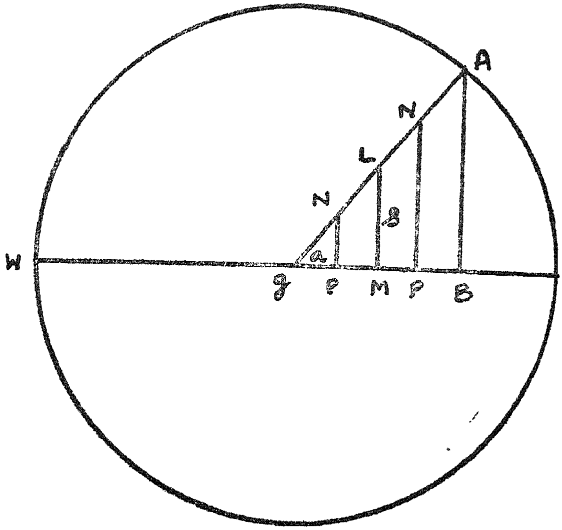


Fig. 45

and that passing through the top of the gnomon being parallel planes cut the horizontal plane in parallel straight lines. (This will be also proved analytically subsequently).

Hence LM $= s$. Now from the figure

$$\frac{LM}{gL} = \frac{AB}{gA} = H \sin a \qquad gL \qquad \frac{H \sin a}{Rs} \qquad I$$

This gL is spoken of as Iṣṭa-Drikmandala palabhā because it is the shadow on the equinoctial day in any vertical. LN is the increment in the shadow on account of declination and we have to compute this and correlate gL and LN. For this refer to figs. 46 and 47. In fig. 46, QRT is the equator, so that when the Sun is on the equator on the equinoctial day in the direction given by ZS, ZT is the zenith-distance. Let ZS be the zenith-distance of the Sun in the same direction on any day. From the analogy of finding H sin δ from H sin λ, from this figure

$$\text{H sin SR} = \frac{\text{H sin ST} \times \text{H sin T}}{\text{R}} \quad \text{II and}$$

$$\text{H sin } \varphi = \frac{\text{H sin ZT} \times \text{H sin } \widehat{\text{T}}}{\text{R}} \quad \text{III so that}$$

$$\frac{\text{SR}}{\text{H sin } \varphi} = \frac{\text{H sin ST}}{\text{H sin ZT}} \therefore \text{H sin ST} = \frac{\text{H sin SR}}{\text{H sin } \varphi}$$

Noting that SR = δ and putting ST = D

$$\text{H sin D} = \frac{\text{H sin } \delta}{\text{H sin } \phi} \times \text{H sin ZT.}$$

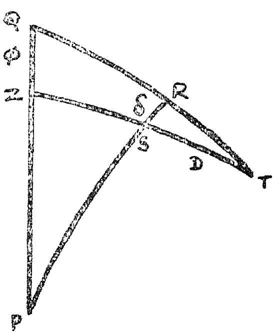


Fig. 46

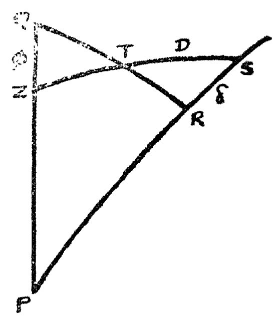


Fig. 47

The same formulae are derivable from fig. 47 also; only in fig. 46 while there is a decrement in the shadow of the day as compared with the shadow on the equinoctial day, in fig. 47, there is an increment. This is seen from the decrease and increase of ST in the zenith-distance ZT of the equinoctial day in the given direction. Now correlating fig. 45 with figures 46 and 47, the shadow *g*L pertains to the zenith-distance ZT on the equinoctial day whereas the shadows gN pertains to the zenith-distance on the day concerned in the same direction. We have,

$$\text{S} \qquad \text{H sin } z \text{ so that } \frac{\text{RS}}{\sqrt{12^2 + \text{S}^2}} = \text{H sin } z \text{ where}$$

S is the shadow at any instant when the zenith-distance is z. The process indicated by saying 'Obtain H sin φ

construing $\frac{Rs}{H \sin a}$ as the equinoctial shadow', means computing H sin ZT and there from ZT from the shadow gL of fig. 45. Then the process indicated by saying "Obtain $H \sin D = \frac{H \sin ZT \times H \sin \delta}{H \sin \varphi}$ and therefrom D" means computing ST.

Then clearly $ZS = ZT \pm ST$, ie. the required zenith-distance is got by what is technically called Samskāra between ZT and ST as is stipulated between φ and δ to obtain Z from the formula $Z + \delta = \varphi$ (The word Samskāra was defined as meaning addition when the directions are the same and difference when they are opposite). Then the gnomonic shadow is got from this zenith-distance using the formula $S = \frac{12 H \sin z}{H \cos z}$.

Thus the procedure adopted by Bhāskara was conceived by him first having Fig. 45 before him and then using figures 46 and 47. In this particular process, H sin a is given and H sin δ also, which means that it is sought to find the shadow on a given day in a given direction.

Incidentally we shall find the locus of the extremity of the gnomonic shadow during the course of a day. Let in fig. 48 g represent the gnomon's foot, and S the shadow whose extremity is p. Required to find the locus of p. Take the gnomon to be of unit length so that the length of the shadow $S = 12 \tan z$ becomes tan z here. Take Eω and sn the east-west line and the north-south as the axes. Then we have

$x^2 + y^2 = \tan^2 z$ (1). But we have from the triangle PZS $\sin \delta = \sin \varphi \cos z + \cos \varphi \sin z \sin a$

ie. $\frac{\sin \delta}{\cos \varphi \cos z} = \tan \varphi + \tan z \sin a = \tan \varphi + y$ (2)

ie. $\frac{\sec z}{A} = y + \tan \varphi$ when A = sin

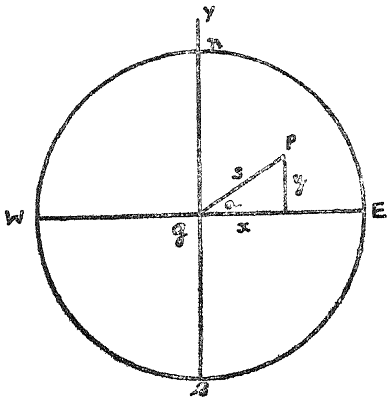


Fig. 48

But $\sec z = \sqrt{1 + \tan^2 z} = \sqrt{x^2 + y^2 + 1}$

$$\therefore \quad \frac{\sqrt{x^2 + y^2 + 1}}{A} = y + \tan \varphi$$ which reduces to

$x^2 + y^2 (1 - A^2) - 2A^2 y \tan \varphi + 1 - A^2 \tan^2 \varphi = o$ (3)
From this it is evident that the locus is an ellipse or parabola or hyperbola according as $A \lesseqgtr 1$; also it will be seen that the eccentricity is A. The locus is wrongly stated to be always a hyperbola in some text books. For it to be an ellipse $A < 1$ ie. $\cos \varphi < \sin \delta$ ie. $\delta > 90 - \varphi$ ie. $\varphi + \delta > 90$. In such latitudes and under such declinations, it will be an ellipse ie. at a place just north of the place where the perpetual day just begins the locus will be an ellipse. Hence in the arctic region it will be always an ellipse; and in the place just at which the perpetual day begins it will be a parabola and in the lower latitudes it will be a hyperbola, ie. it will be a parabola where the latitude φ is given by $90 - \delta$.

When $\varphi = 90°$, $A =$ $= o$ provided $\delta \neq o$. If, however, in addition $\delta = o$, A becomes indeterminate, but we may note then, that the Sun will be circling round the horizon on that equinoctial day at the north pole. We may further note that at the north pole, the altitude of the Sun is always δ so that the length of the shadow cast is always equal to cot δ and this will be infinite when $\delta = o$.

If now $\phi = 90°$, and $\delta \neq o$, though $A = \frac{\cos \phi}{\sin \delta}$ becomes zero, A tan ϕ will not be zero because A tan ϕ = $\frac{\cos \phi}{\sin \delta} \times \tan \phi = \frac{\sin \varphi}{\sin \delta} = \frac{1}{\sin \delta}$ ($\therefore \varphi = 90°$). On the other hand $A^2 \tan \varphi = A \times A \tan \varphi = \frac{1}{\sin \delta} = O$ because $A = o$. Thus the term containing y in eqn. (3) vanishes.

$\therefore$ The equation reduces to

$x^2 + y^2 = A^2 \tan^2 \phi - 1 = \text{cosec}^2 \delta - 1 = \cot^2 \delta$ (ie. +ve) ie. at the north pole the locus will be a circle with radius cot δ. When

$A = \infty$ the locus $\frac{\sqrt{x^2 + y^2 + 1}}{A} = y + \tan \phi$ becomes $y = -\tan \phi$ which means that the hyperbola degenerates into the straight line which is parallel to the east-west line and is in the north at a distance of tan ϕ ie. s, the equinoctial shadow since the length of the gnomon is taken to be unity. In particular when A tan ϕ = 1 ie. $\varphi = \delta$, the constant in (3) is zero, so that the locus passes through the foot of the gnomon as is also evident from the fact that the Sun passes through the zenith.

Taking a northern latitude say 17°, the loci of the extremity of the shadow are shown in fig. 48A (page 270) on important days when $\delta = \omega$, when $\delta = o$, when $\delta = -\omega$, when $\delta = \varphi$. The maximum mid-day shadow is $\tan (\varphi + \omega)$, when $\delta = -\omega$ taking the gnomon's length to be unity; this shadow is cast north of the gnomon along the south-north line through the gnomon, on Dec. 23rd of the year. The minimum length of the mid-day shadow occurs when $\delta = \varphi$, the shadow being zero and being at the foot of the gnomon, the Sun being then just overhead. The maximum shadow cast south of the gnomon at mid-day is $\tan (\omega - \varphi)$.

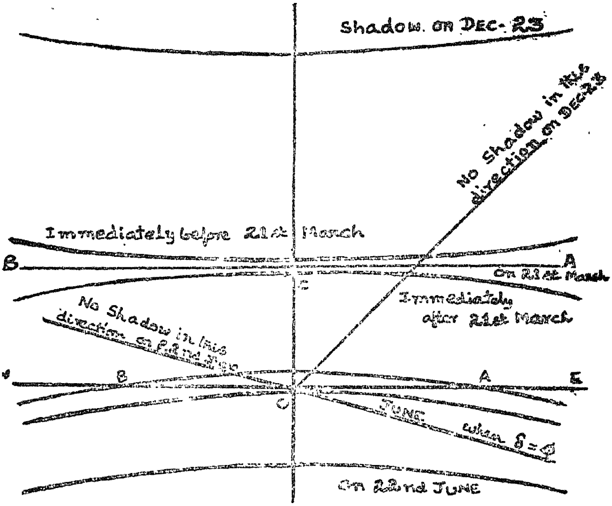


Fig. 48A Showing the locus of the extremity of the shadow on different days at a latitude of 17°

Note. OA, OB are the shadows computed by Bhāskara when the Sun is on the prime-vertical.

In the method of finding the shadow under verse 46, we perceive Bhāskara's genius in (1) looking upon the shadow as being made up of two segments namely that due to φ and that due to the declination (2) in conceiving what he calls Iṣṭa-drik-mandala palabhā, and Iṣṭa-drik-mandala Krānti and (3) in deriving the equation

$$\text{H sin D} = \frac{\text{H sin ZT} \times \text{H sin } \delta}{\text{H sin } \varphi}$$

Latter half of verse 47 *and verse* 48. Something to be noted.

In computing the shadow in a given direction, there may be two shadows at times in the northern hemisphere. When H sin a < Agrā and there will be none in the southern. To compute the second shadow we have to take 180 − L also as the latitude where L is the latitude computed, and proceed in the same way as we have done before.

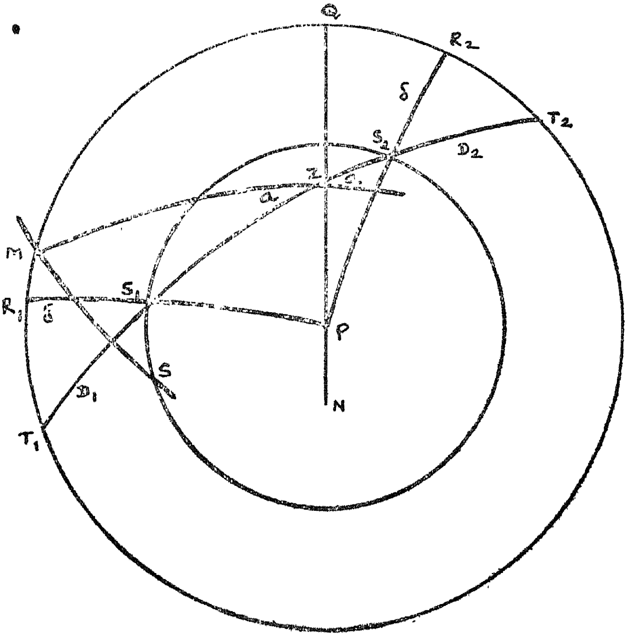


Fig. 49

Comm. This too exhibits Bhāskara's genius. (Ref. fig. 49). Let MQR_2 be the equator whose pole is p. Let $T_1 S_1 Z S_2 T_2$ be the circle of azimuth a (Hindu azimuth). Let $SS_1 S_2$ be the diurnal circle of the Sun cutting the above circle of azimuth at S_1 and S_2, so that ZS_1 and ZS_2 are the two solutions giving the two zenith-distances which give two shadows in the given direction. H sin MS = Agrā ; evidently $\widehat{MZS} > \widehat{MZS_1}$ ie. H sin $a <$ Agrā as stipulated. ZT_1 and ZT_2 give the zenith-distances in the given direction when the Sun is on the equator. $S_1 T_1$ and $S_2 T_2$ are the decrements in the zenith-distances on account of declination δ (= $S_2 R_2$ or S_1 If MZS_1 were greater than $\widehat{MZS}$ ie. if H sin $a >$ Agrā, we would have lost the position S_1 ie. we would have had only one shadow in the afternoon in the given direction and no shadow in the morning.

Analytically, the event of having two shadows arises on account of the following circumstance. When we are asked to find Iṣṭākṣajyā from $\frac{Rs}{H \sin a}$ the shadow in the given direction on the equinoctial day (Ref. gL fig. 45) the L for this given value of the shadow is given by

H L = where S = $\frac{Rs}{\cdot}$. We know

$\sin \theta = a$ has two solutions, θ_1 and $180 - \theta_1$. Hence L will have two values L_1 and $180 - L_1$. So, Bhāskara has asked us to compute D_1 and D_2 from L_1 and L_2 and thus have the two solutions.

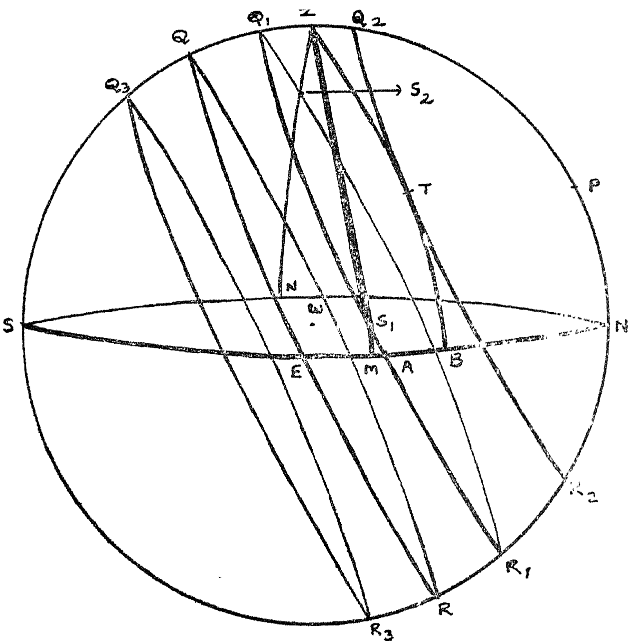


Fig. 50

Note (1) When Bhāskara said 'If H sin a < Agrā' he had in mind evidently the azimuth circle MZN which cuts the diurnal path A Q_1 R_1 at S_1 and S_2. At S_1 the azimuth EZM < EZA so that he stipulated that H sin a should be less than the Agrā. But, let the diurnal path of the Sun be Q_2 R_2 where Q_2 falls in between z and p. In such a case we know that the azimuth does not take all values but has a maximum where the vertical touches the diurnal path at T. From PTZ where T is a right angle, taking PZT = 90 − a, a being the Hindu azimuth we have by Napier's rule sin PT = sin ZP sin (90 − a) or

$\sin(90-\delta) = \sin(90-\varphi)\sin\overline{90-a}$ or $\cos\delta = \cos\varphi\cos a$. If a has a lesser value than is given by this equation, the diurnal path does not cut the azimuth circle ie. if $\cos a > \dfrac{\cos\delta}{\cos\varphi}$, there will be no shadow in the given direction even though the situation satisfies Bhāskara's condition namely H sin a should be less than Agrā. *Bhāskara has overlooked this case.* This may be seen analytically also as follows. We have from the spherical triangle PZS, $\sin\delta = \sin\varphi\cos z + \cos\varphi\sin z\sin a$ $= A\cos z + B\sin z$ (say). We know, the maximum value of $A\cos z + B\sin z$ is $\sqrt{A^2 + B^2}$ which is here $\sqrt{\sin^2\varphi + \cos^2\varphi\sin^2 a} = \sqrt{\sin^2\varphi + \cos^2\varphi - \cos^2\varphi\cos^2 a}$ $= \sqrt{1 - \cos^2\varphi\cos^2 a}$. Thus there will be no solution for z if the quantity on the left hand side namely

$\sin\delta >$ the above max. value ie. if

$\sin\delta > \sqrt{1-\cos^2\varphi\cos^2 a}$ ie. if $\sin^2\delta > 1-\cos^2\varphi\cos^2 a$

ie. if $\cos^2\delta > \cos^2\varphi\cos^2 a$ ie. if $\cos\delta > \cos\varphi\cos a$

ie. if $\cos a > \cos\delta/\cos\varphi$ as derived above.

Hence even if H sin a > Agrā, there need not be a shadow at all in the given direction. In other words when the Hindu azimuth given is very small and when the declination is too great north or south, there may not be a shadow in the given direction. Bhāskara gives an example where he gets two shadows on a day taking the moments when the Sun is on the prime-vertical. In fact having this case of the East-West shadows alone, he conceived that two shadows could be had in a given direction under particular conditions.

He chooses a place of $s = 5''$ ie. a place of latitude $22° - 37'$ (Bhāskara often gives this latitude which might indicate that he was probably residing in that latitude which passes through approximately Itarsi). He takes a day when the Sun's declination is given by $H \sin \delta = 780$ ie. $\delta = 13° - 7'$. Then the Sama-S'anku is given by $\frac{R \sin \delta}{\sin \varphi}$ (comparing the second and the fifth latitudinal triangles

$$\frac{\text{Sama-S'anku}}{R} = \frac{\text{Krantijyā}}{H \sin \varphi} = \frac{H \sin \delta}{H \sin \varphi}$$

$$\therefore \text{ Sama-S'anku} = \frac{RH \sin \delta}{H \sin \varphi} = \frac{t}{\sin \varphi}$$

$$= \frac{3438 \times 780}{} = 2028 \text{ approximately. I}$$

$$\text{Also Agrā} = \frac{R \sin \delta}{\cos \varphi} = 845.$$

Knowing the Sama-S'anku, the East-West shadow may be taken to be determined. Now Bhāskara proceeds to show that at the time of having the second shadow also, in the same East-West direction, the S'anku will be the same Sama-S'anku itself. For this, proceeding according to the method indicated in the verse, " Taking $\frac{Rs}{H \sin a}$ to be the equinoctial shadow etc." we have

$$\frac{Rs}{H \sin a} = \frac{3438 \times 5}{o} = \text{what is called Kha-hara Rās'i.}$$

Taking this to be the equinoctial shadow

$\text{H sin L} = \frac{Rs}{\sqrt{12^2+s^2}}$ = R itself (Dealing this way with Kha-hara Rās'is is prohibited in modern mathematics but Bhāskara adds at the end of the commentary that dealing with them cautiously does not effect computations which is of course true, for when the equinoctial shadow is infinity $\varphi = 90°$ so that $\text{H sin } \varphi = R$ as got). Hence $L = 90°$ and $180° - 90° = 90° = L'$. Then $\text{H sin D} = \frac{R \times 780}{1322 - 18}$ = same as Sama-S'anku obtained in $I = \text{H cos z}$ so that $D = 90 - z$. Now from the equation $z + \delta = \varphi$, $z = \varphi - \delta = L' - D = 90° - (90 - z)$ where z is the zenith-distance when the Sun is on the prime-vertical.

∴ The zenith-distance is again the same z. In other words, the second zenith-distance is also that when the Sun is on the prime-vertical. Bhāskara has given this example just to obtain the second shadow as well and he has chosen the event of the Sun being on the prime-vertical to show that the procedure indicated by him may be verified to hold good.

Verses 49, 50. Alternate method to find the shadow.

Let $R^2 s^2 + \text{H sin}^2 a \times 12^2$ = prathama where s = equinoctial shadow and a the Hindu azimuth. Let Anya = RsA where A is the Agrā. Divide the prathama and Anya by ($\text{H sin}^2 a - A^2$) and still call them prathama and Anya. Then

$K = \sqrt{\text{Adya} + \text{Anya}^2} \pm \text{Anya}$ where K is the Chāyā-

Comm. Let K be the required Chayā-Karṇa.

Then Karnāgrā $= \frac{KA}{R} = s + b$ where b is the bhuja.

$= \frac{KA}{R} - s$. But $\frac{H \sin a}{b} = \frac{R}{S}$ where S is the shadow so that

$$S = \frac{bR}{H \sin a} = \left(\frac{KA}{R} - s\right) \frac{}{H \sin a}$$

$$\therefore \; R(KA - sR) = RH \sin a \sqrt{K^2 - 12^2}$$

$$\text{ie. } R^2 (K^2 A^2 + s^2 R^2 - 2 AsRK = H \sin^2 a \, (K^2 - 12^2)$$

$$\therefore \; K^2 (A^2 - H \sin^2 a) - 2 AsRK = - s^2 R^2 - 12^2 H \sin^2 a$$

$$\therefore \; K^2 (H \sin^2 a - A^2) + 2 AsRK = 12^2 H \sin^2 a + s^2 R^2 \quad \text{I}$$

This quadratic is of the form $ax^2 + 2bx = c$

ie. $x^2 + 2\frac{b}{a}x = c/a$ II Here $\frac{`c'}{a}$ is called Adya and $\frac{b}{a}$ Anya.

The solution of the above equation is given by $x = -\frac{b}{a} \pm \sqrt{\frac{b^2}{a^2} + c/a}$

ie. $-$ Anya $\pm \sqrt{\text{Anya}^2 + \text{Adya}}$. III

When $b = \frac{KA}{R} + s$, putting $-s$ in the place of s in I,

K = Anya $\pm \sqrt{\text{Anya}^2 + \text{Adya}}$ IV

Out of the four solutions given by III and IV we have taking the positive solutions

K = $\sqrt{\text{Anya}^2 + \text{Adya}} \pm$ Anya as stated.

Verse 51. If H sin $a <$ A, then in the northern hemisphere ie. where δ is north,

$\pm \sqrt{\text{Anya}^2 - \text{Adya}}$ + Anya = K.

Comm. We have initially put Anya = H $\sin^2 a - A^2$. If H sin $a <$ A, them to avoid a negative value for the

Anya, we could put Anya $= A^2 - H\sin^2 a$. As a matter of fact in verse 50, we are asked to take $H\sin^2 \sim A^2$, as Bhāskara wanted that the second case also be included.

Thus putting Anya$=A^2-H\sin^2 a$, equation I becomes $K^2 (A^2 - H\sin^2 a) - 2\,AsRK = -$ (Adya) so that
$K =$ Anya $\pm \sqrt{\text{Anya}^2 - \text{Adya}}$ as given.

Verse 52. The Bhuja is to be obtained through Karṇāgra from the equation $a = b + s$ (already proved) and $\frac{Rb}{S} = H\sin a$ $\{$ie. $S = \frac{bR}{H\sin a}$ as already proved$\}$.

This H sin a will be the same in the case of obtaining two values of K ie. two shadows one in the morning and the other in the afternoon, (the only difference being that they will be on alternate sides of the East-West line).

Verses 53 *and* 54. Obtaining the shadow when time is given.

In the two previous examples the magnitude of the shadow was obtained in a given direction; now we shall obtain the same when time is given. The word unnata stands for the time that has elapsed after Sun-rise or that which is the balance of the day time. The unnata subtracted from half-the-day gives what is called Nata.

The H sine of the unnata minus Chara or increased by the Chara according as the Sun's declination is north or south, is called Sūtra; this multiplied by the H cos δ and divided by the radius, is called Kalā.

Comm. (Ref fig. 51) The time measured by the arc MN, that is the time in between the moment when the Sun S is on the horizon and the moment when he is at L is called the unnata ie. the time measured after rising and the time measured by the arc NQ ie. the time in

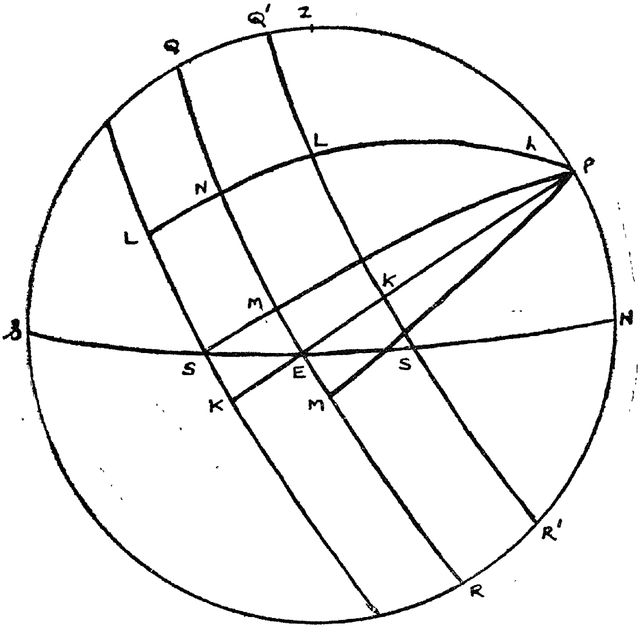


Fig. 51

the moments when the Sun is at L and when he reaches the meridian. is called Nata. In modern astronomy this Nata is known as the hour angle h and unnata $= H - h$ where H is the rising hour angle. The time measured by the arc ME is called Chara. Thus unnata-chara $=$ EN and H sin EN H sin $(90 - h) = H \cos h$ is called Sutra which is BN shown in fig. 52 representing the Equator. The corresponding line bn in fig. 53 which represents the diurnal circle, is known as Kalā. Thus Sūtra$=H \cos h$ (26) and Kalā $= \dfrac{H \cos h \times H \cos \delta}{R}$ (27).

When the Sun is in the Southern hemisphere, unnata is measured by the arc SL or MN ie. $(H - h)$ where H is

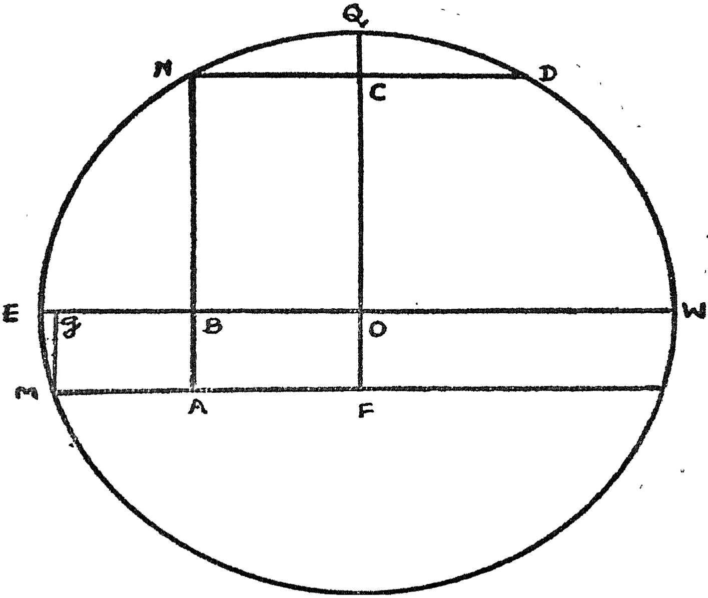


Fig. 52

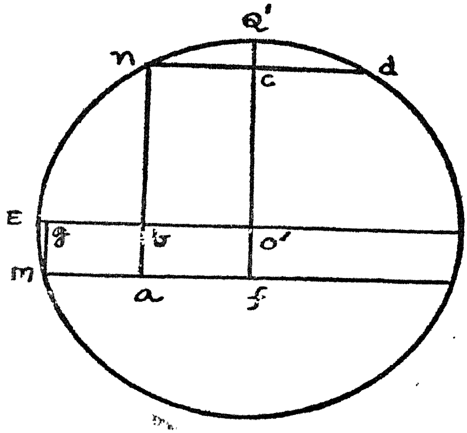


Fig. 53

the rising hour angle and $h = \widehat{LPQ}$ and Chara by the arc EM and Sūtra = H sin (MN + EM) = H sin (unnata + chara) = H sin EN = H sin (90 − h) = H cos h.

Verse 55. Sūtra multiplied by Kujyā and divided by Charajyā will be also Kalā and Kalā multiplied by the Koti and divided by the Karṇa of any latitudinal triangle will be Iṣta *yaṣṭi*.

Comm. In as much as Kalā is the corresponding line in the diurnal circle to the Sūtra in the plane of the celestial equator

$$\frac{\text{Sūtra}}{\text{Kalā}} = \frac{\text{R}}{\text{H cos } \delta} = \frac{\text{Charajyā}}{\text{Kujyā}} \quad \therefore \frac{\text{Sūtra} \times \text{Kujyā}}{\text{Charajyā}} = \text{Kalā}$$

In verse 33, we saw that

Dinārdha S'anku − Unmandala S'anku = Yaṣṭi. This Iṣta-yaṣṭi will be therefore the perpendicular dropped from n, the Sun's position in the diurnal circle on the plane parallel to the horizon and passing through the head of the Unmandala S'anku ie. passing through B and parallel to the horizon in fig. 21. Since the angle between the diurnal plane and the vertical plane of the yaṣṭi is equal to φ the latitude, the Iṣtayaṣṭi forms a latitudinal triangle with the Kalā, it being the Koti or side opposite to the angle 90 − φ and the Kalā being the hypoten use,

$$\therefore \quad \frac{\text{Yaṣṭi}}{\text{Kalā}} = \cos \varphi = \frac{\text{Koti of a latitudinal triangle}}{\text{Karṇa of a latitudinal triangle}}$$

$$\therefore \quad \text{Iṣtayaṣṭi} = \text{Kalā} \times \frac{\text{Koti of a latitudinal triangle}}{\text{Karṇa}}.$$

When the Sun is on the meridian, Iṣtayaṣṭi becomes yaṣṭi of verse 33.

The formula for Iṣtayaṣṭi is therefore

$$\frac{\text{Kalā} \times \text{H cos } \varphi}{\text{R}} = \frac{\text{H cos } \delta \ \text{ H cos } h}{\text{R}} \times \frac{\text{H cos } \varphi}{\text{R}}$$

from formula (27)

$$= \frac{\text{H cos } \varphi \text{ H cos } \delta \text{ H cos } h}{\text{R}^2} \qquad 27'$$

First half of verse 56. The Unmandala S'anku multiplied by the Sūtra and divided by Charajyā is also Iṣṭayaṣṭi.

Comm. The Unmandala S'anku and Iṣṭayaṣṭi are the lines in vertical planes corresponding to Charajyā and Sūtra in the Equatorial plane. Hence the proportion. It will be noted that the Unmandala S'anku and Iṣṭayaṣṭi are not in the same vertical plane but parallel vertical planes. None the less the proportionality holds good.

Latter half of verse 56 *and first half of verse* 57.

The Sūtra increased or decreased by the Charajyā according as the Sun is in the northern or southern hemisphere is what is known as Iṣṭāntyā; similarly the Kalā increased or decreased by Kujyā is what is known as Iṣṭa-Hṛti.

Comm. In fig. 52, Iṣṭāntyā = AN = AB + BN = ME + BN = Charajyā + Sūtra. Similarly in fig. 53, Iṣṭa Hṛti = $an = ab + bn = sg + bn$ = Kujyā + Kalā.

$$\therefore \text{ Iṣṭāntyā} = \text{H} \cos h + \text{R} \tan \delta \tan \varphi =$$

$$\text{R} \frac{(\sin \varphi \sin \delta + \cos \varphi \cos \delta \cos h)}{\cos \varphi \cos \delta} \text{ in modern terms} \quad (28)$$

Latter half of verse 57. Similarly Iṣṭayaṣhṭi increased or decreased by the Unmandala S'anku is Iṣṭa S'anku or H cos z.

Comm. Let in fig. 54 which represents the plane of the prime-vertical AA′, EW, BB′, FF′, qq′ represent the lines of intersection of this plane with planes parallel to the horizon and passing through A, B, F, q of fig. 21. Then $O\alpha$ = Unmanda-S'anku, $O\beta$ = Sama S'anku, $O\gamma$ = Dinārdha S'anku. If xx' be the line of intersection of this plane of the prime-vertical with a plane passing through an arbitrary position of the Sun in the diurnal circle and

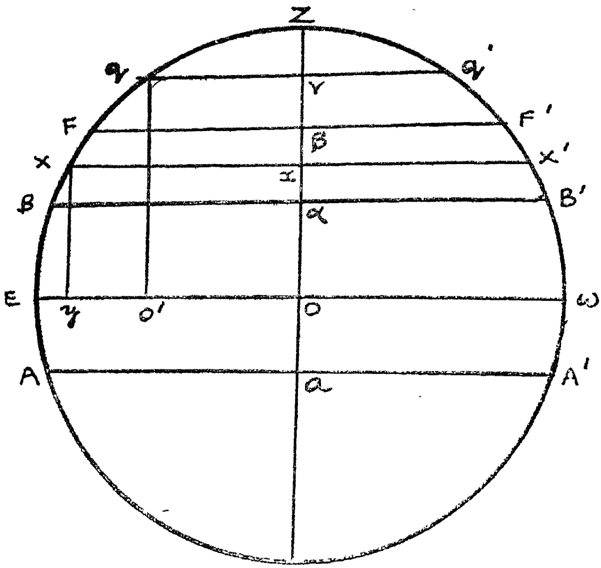


Fig. 54

parallel to the horizon. then Ox = Iṣṭa-S'anku = Oa + ax = Un-mandala S'anku + Iṣṭa yasti. ar = yasti.

In the Southern hemisphere Oa, the Unmandala S'anku will be below the horizon so that Iṣṭayaṣṭi decreased by the Unmandala S'anku will be Iṣṭa-S'anku.

Thus we have the method of obtaining the Iṣhta-S'anku from the Unnata Kāla as detailed above. We shall see what this process means in modern terms.

Unnatakāla-Charakāla = $\odot\widehat{PA} - \widehat{APE}$ (Fig. 21) = $\odot\widehat{PE}$ where $\odot$ is the foot of the declination circle of the Sun e in any arbitrary position in his diurnal path. But $\odot PE = \widehat{QPE} - \widehat{QP\odot} = 90-h$ where h is the hour an Sun. Thus Sutra = Ḥ sin $(90-h)$ = Ḥ

$$\therefore \quad \text{Kalā} = \frac{\text{H}\cos h \times \text{H}\cos\delta}{\text{R}} \qquad \text{II}$$

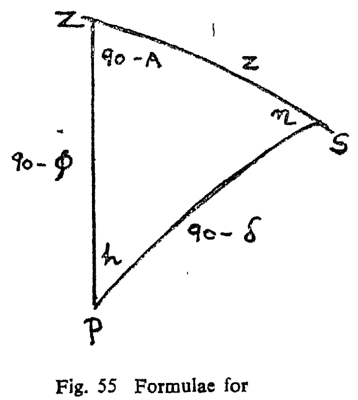
$$\therefore \quad \text{Iṣṭa-yaṣṭi} = \frac{\text{H}\cos h \times \text{H}\cos\delta}{\text{R}} \times \frac{\text{H}\cos\varphi}{\text{R}} \qquad \text{III}$$

Now Unmandala-S'anku is derivable from the sixth latitudinal triangle in which Krāntijyā is the Karṇa and Unmandala-S'anku is the Bhuja. Comparing it with the second latitudinal triangle $\frac{\text{Krāntijyā}}{\text{R}} = \frac{\text{U. S.}}{\text{H}\sin\varphi}$ where U. S. is Unmandala-S'anku.

$$\therefore \quad \text{U. S.} = \frac{\text{Krāntijyā} \times \text{H}\sin\varphi}{\text{R}} = \frac{\text{H}\sin\varphi\ \text{H}\sin\delta}{\text{R}} \qquad \text{IV}$$

∴ Iṣta-S'anku as per the above formulation is given by Iṣta-S'anku (I. S.) =

$$\frac{\text{H}\sin\delta\ \text{H}\sin\delta}{\text{R}} + \frac{\text{H}\cos\varphi\ \text{H}\cos\delta\ \text{H}\cos h}{\text{R}^2} = \text{I}$$



(Ref. fig. 55) Fig. 55 Formulae for

Z = zenith ; P = celestial pole. S = celestial body, say, the Sun.

= zenith-distance of the celestial body S.

PZ = colatitude ; PS = north-polar-distance or co-declination.

n is called the parallactic angle ; h = hour-angle of S. $(90 - a)$ = The complement of the Hindu azimuth a being measured from the East point.

$$\cos(90-\delta) = \cos(90-\varphi)\cos z + \sin(90-\varphi)\sin z\cos(90-a)$$

$$\delta = \sin\varphi + \cos z + \cos\varphi\sin z\sin a \quad (1)$$

$$\cos z = \cos(90-\varphi)\cos(90-\delta) + \sin(90-\varphi)\times \sin(90-\delta)\cos h$$

$$= \sin\varphi\sin\delta + \cos\varphi\cos\delta\cos h \quad (2)$$

$$\cos(90-\varphi)\cos(90-a) = \sin(90-\varphi)\cot z - \sin(90-a)\cot h$$

ie. $\sin\varphi\sin a = \cos\varphi\cot z - \cos a\cot h$ (3)

$$\cos(90-\varphi)\cos h = \sin(90-\varphi)\cot(90-\delta) - \sin h \times \cot(90-a)$$

ie. $\sin\varphi\cos h = \cos\varphi\tan\delta - \sin h\tan a$ (4)

$$\frac{\sin a}{\sin(90-a)} = \sin h\cos\delta \quad (5)$$

This means in modern terms

$\cos z = \sin\varphi\sin\delta + \cos\varphi\cos\delta\cos h$ V which we derive from the triangle PZS.

58. To get H cos z from h the hour-angle or nata Kāla.

The H. vers (Nata) is called S'ara (CQ of fig. 52) (29)

Antyā — S'ara = Iṣṭāntyā ie. FQ — CQ = FC = AN (fig. 51)

(30)

Hṛti — phala = Iṣta-Hṛti ie. $fq - cq = fc = an$ (fig. 53)

Comm. (Ref. fig. 52). Nata = arc QN

$\therefore$ H. vers (Nata) = QC. (Called S'ara) = H. vers (h)

Antyā — S'ara = FQ — CQ = FC = AN = Iṣtāntya II

S'ara $\times \frac{\text{H} \cos \delta}{\text{R}}$ = phala = cq (fig. 53) = $\frac{\text{Hvers } h \times \text{H} \cos \delta}{\text{R}}$

Hṛti — phala = $fq - cq = fc = an$ = Iṣta-hṛti IV

3 59. $\frac{\text{Phala} \times \text{Koti of a latitudinal triangle}}{\text{Karṇa}}$

= Ūrdhwa V

= βr (of fig. 54)

Comm. (Ref. fig. 54) βr is the corresponding line in the plane of the meridian corresponding to phala in the plane of the diurnal circle. As the angle between these two planes is the latitude itself, by the principle of orthogonal projection namely.

Magnitude of a projected segment = cosine of the dihedral angle $\times$ the magnitude of the segment projected, since Ūrdhwa is the orthogonally projected segment of phala, so, Ūrdhwa = phala $\times \cos \varphi$

= phala $\times$ H cos φ VI

= $\frac{\text{phala} \times \text{Koti of a latitudinal triangle}}{\text{Karna of the latitude triangle}}$ as stated

Thus Ūrdhwa = βr (of fig. 54).

30. Ūrdhwa is also given by

Ūrdhwa = $\frac{\text{U.S. (Unmandala S'anku)} \times \text{S'ara}}{\text{Charajyā}}$.

Dinārdha-S'anku (D.S.)—Ūrdhwa = Iṣta-S'anku (I.S.). H cos z.

Comm. Since U.S. is the projected segment of Charajyā on a vertical plane and since the diahedral angle between the planes is φ the latitude

$$\frac{\text{U.S.}}{\text{Charajyā}} = \text{Cos } \varphi \text{ and so S'ara} \times \cos \varphi = \frac{\text{S'ara} \times \text{H} \cos \varphi}{\text{R}}$$

= Ūrdhwa.

From fig. 54, Dinārdha-S'anku − Ūrdhwa = $o'q - \beta r$ (fig. 54) = $o'\beta' = y\text{X}$ = Iṣta-S'anku.

In modern times, this means,

S'ara = H. vers (h) = (R − cos h) = 1 − cos h in modern terms

$$\text{phala} = \frac{\text{S'ara} \times \text{H} \cos \delta}{\text{R}} = (1 - \cos h) \times \cos \delta \quad ,,$$

$$\text{phala} \times \frac{\text{H} \cos \varphi}{\text{R}} = \text{Ūrdhwa} = (1 - \cos h) \cos \delta \times \cos \varphi \;,, \;,,$$

Dinārdha S'anku − Ūrdhwa = Iṣta S'anku H cos z
= cos z (in modern terms)

= H cos ($\varphi - \delta$) (taking northern declination and following Hindu convention with respect to signs)
= Dinārdha-S'anku
= cos $\varphi - \delta$ in modern terms.

$\therefore \cos(\varphi - \delta) - (1 - \cos h) \cos \varphi \cos \delta = \cos z$

ie. $\cos \varphi \cos \delta + \sin \varphi \sin \delta - \cos \varphi \cos \delta + \cos \varphi \cos \delta \cos h = \cos z$

ie. $\cos z = \sin \varphi \sin \delta + \cos \varphi \cos \delta \cos h$ as before.

Verse 61. Computation of H cos z (Mahā S'anku) through Antyā and Hṛti.

Let the Dinārdha S'anku (D.S.) be computed through Iṣtāntyā and Iṣta Hṛti and therefrom Iṣta S'anku. From the S'anku, Drik-jyā ie. H sin z and the shadow $\frac{\text{KH} \sin z}{\text{R}}$ could be computed −*H sin z should not be computed* from Hṛti.

Comm. We computed Dinārdha S'anku by the formula.

$$D.S. = \frac{\text{Antyā} \times \text{U.S.}}{\text{Charajyā}} = \frac{\text{Hṛti} \times \text{Koti of a L.T.}}{\text{Karṇa of L.T.}}$$

under verse 36 ; similarly

Iṣta S'anku (I.S.) will be given by

$$\text{Iṣta S'anku} = \frac{\text{Iṣtāntya} \times \text{U.S.}}{\text{Charajyā}} = \frac{\text{Iṣta Hṛti} \times \text{Koti of a L.T.}}{\text{Karṇa of the L.T.}}$$

$$\text{But Iṣtāntyā} = \frac{R\,(\sin\varphi \sin\delta + \cos\varphi \cos\delta \cos h)}{\cos\varphi \cos\delta}$$

(as under verse 56)

$$\therefore \text{ Iṣta S'anku} = \frac{R\,(\sin\varphi \sin\delta + \cos\varphi \cos\delta \cos}{\cos\varphi \cos\delta}$$

$$\times \frac{R \sin\delta \sin\varphi}{R \tan\varphi \tan\delta} \text{ from formulae (13) and (19)}$$

$$= R\,(\sin\varphi \sin\delta + \cos\varphi \cos\delta \cos h) = R \cos z = H \cos z$$

$$\text{Or again Iṣta Hṛti} = \frac{R \cos z}{\cos\varphi} \text{ from formula (11)}$$

$$\therefore \text{ Iṣta S'anku} = \frac{R \cos z}{\cos\varphi} \times \frac{H \cos\varphi}{R} = R \cos z = H \cos z$$

Having got H cos z, using the formula $H \sin^2 z = R^2 - H \cos^2 z$, H sin z ie. Dṛk-jyā can be computed. Also $K = \frac{12R}{H \cos z}$ and $S = \frac{KH \sin z}{R}$ give the Chāyā Karṇa and Chāyā.

Bhāskara cautions us that H sin z cannot be computed from Hṛti as mentioned in verse 37. because there in that verse, the H sin z computed is that at noon alone.

Verse 62. Alternate method of obtaining K.

The Chāyākarṇa when the Sun is on the unmandala multiplied by Kujyā or that when the Sun on the prime-vertical multiplied by Taddhṛti, or again that when the Sun is on the meridian multiplied by Hrti, divided by Iṣta Hṛti, will be equal to the Iṣta-Karṇa K.

Comm. Equation (23) under verse 41 is

$$\frac{\text{Iṣṭa S'anku}}{\text{Iṣṭa Hṛti}} = \frac{\text{D.S.}}{\text{Hṛti}} = \frac{\text{S.S.}}{\text{Taddhṛti}} = \frac{\text{U.S.}}{\text{Kujyā}} = \cos\varphi \quad \text{I}$$

and $\frac{12}{K} = \frac{H\cos z}{R}$ (under verse 40)

$$\frac{12R}{\text{Iṣṭa S'anku}} \text{ which means}$$

Dinārdha Karṇa $= \frac{12R}{D.S.}$; Sama Karṇa $\frac{12R}{S.S.}$

and unmandala Karṇa $= \frac{12R}{U.S.}$.

Substituting for the numerators in I

$$\frac{12R}{K \times \text{Iṣṭa Hṛti}} = \frac{12R}{DK \times \text{Hṛti}} = \frac{12R}{S.K \times \text{Taddhṛti}}$$

$$= \frac{12R}{U.K. \times \text{Kujyā}} \quad \text{II}$$

$\therefore$ Iṣṭa Karṇa $\times$ Iṣṭa Hṛti $=$ *S.K.* $\times$ Taddhṛti

$=$ U.K. Kujyā

$$\therefore \text{Iṣṭa Karṇa} = \frac{\text{Uumandala Karṇa} \times \text{Kujyā}}{\text{Iṣṭa Hṛti}}$$

$$= \frac{\text{Sama Karṇa} \times \text{Taddhṛti}}{\text{Iṣṭa Hṛti}} = \frac{\text{Dinārdha Karṇa} \times \text{Hṛti}}{\text{Iṣṭa Hṛti}}$$

Verse 63. Just a caution.

If in any context where the word ūna-yuta has been used, the quantity to be subtracted exceeds the quantity from which it is to be subtracted, it goes without saying that subtraction should be reversely effected and in the place of addition subsequently prescribed subtraction should be done and Vice-versa.

Comm. Bhāskara gives three examples to illustrate his point. In verse 54, while defining Sūtra (H sin $\overline{90-h}$) we are asked to subtract chara from unnata when $\delta > o$

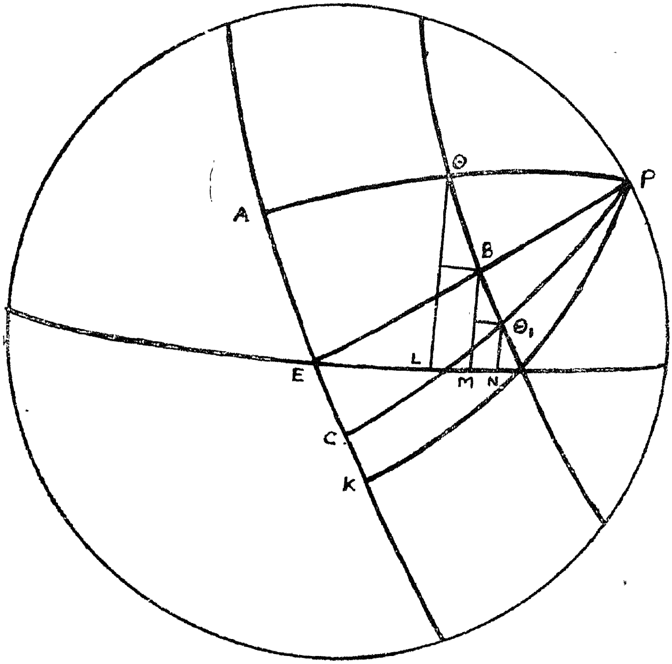


Fig. 56

and add Chara to Unnata when $\delta < o$ and take the H sine of the result. Let us first consider the case when $\delta > o$. (Refer fig. 56) when the Sun is at ⊙, Iṣta S'anku is H sin ⊙L; and Unmandala S'anku is H sin BM. When the Sun is at $⊙_1$, Iṣta S'anku = H sin $⊙_1$N. In the first case Iṣtayaṣti = (H sin ⊙ L — H sin BM) which will be the orthogonal projection of ⊙ B on the meridian plane. Unmandala S'anku and the Iṣta S'anku in the position $⊙_1$ are similarly the orthogonal projections of BM and $⊙_1$N on the same plane. In the position ⊙ Iṣta S'anku = Unmandala S'anku + Iṣtayaṣti, whereas in the position $⊙_1$, Iṣta S'anku = Unmandala S'anku — Iṣtayaṣti which is now downwards. Thus in the place of addition we have subtraction of Iṣtayaṣti. This reversion has arisen out of the fact that in the position ⊙, Sūtra is the H sine of (KA — KE) whereas in the position $\odot_1$ Sūtra is the H sine of EC ie. H sine of (KE — KC) ie. in the former position Sūtra = H sine (Unnata-Chara) and in the lattter Sūtra = H sin (Chara-Unnata). Thus a reversion in subtraction here, effects a reversion of addition of the Iṣṭayaṣṭi.

Similar is the case in the other cases cited by Bhāskara. Analytically this happens so because cos h, when $h > 90$, becomes negative and adding cos h tantamounts to subtracting sin θ where $h = 90 + \theta$.

Verse 64. Another point to be observed.

$$\text{Hvers } (90 + \theta) = R - H \cos \overline{90 + \theta} = R + H \sin \theta.$$

The Unmandala S'anku is not observable when δ is south in as much as it is below the horizon ; none the less it may be computed for the purposes of effecting proportion.

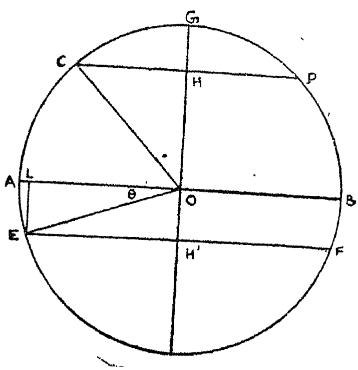


Fig. 57

$$\text{Hvers (CG)} = \text{Hvers } (\widehat{COG}) = GH$$

$$\text{Hvers (EG)} = \text{Hvers } (\widehat{EOG}) = \text{Hvers } (90 + \theta)$$
$$= R + OH^1 = R + EL = R + H \sin \odot$$

as defined by Bhāskara.

Comm. Under verse 58, we had to form Hvers h; a doubt might arise as to what this Hvers h would be if $h > 90$. Hence Bhāskara defines it and the definition is clear from fig. 57. The Unmandala S'anku has the formula R sin δ sin φ so that either when δ is negative or when φ is negative, it will be negative which means it will be in the opposite direction ie. vertically downwards. Since negative latitudes are not considered by the Hindu astronomers, the other case alone is considered. Even from a figure it is evident that when the Sun is in the south of the Equator, the Unmandala S'anku is vertically downwards. In proportions like I given under verse 62, we can use the magnitude of this Unmandala S'anku also and it does not vitiate the results, when we take its numerical value.

Verse 65. The Sun crosses the prime-vertical when his northern declination falls short of the latitude. Then alone there is sense in giving the magnitude of his shadow at that moment. When the Sun does not cross the prime-vertical at all, the Sama S'anku which could be computed out of its formulation, though it does not exist, under the Sun, yet, it could be used in proportions (like I under verse 62) and no blunder is committed.

Comm. This is a beautiful example cited by Bhāskara where he intuitively uses the so-called principle of geometrical continuity. We have formulated Sama S'anku as $\frac{R \sin \delta}{\sin \varphi}$ ie. $H \cos z = \frac{R \sin \delta}{\sin \varphi}$. We have a real value of z when $\delta > \varphi$, for, then only $H \cos z < R$. When $\varphi > \delta$, then $H \cos z > R$ which is impossible, for, no Hindu sine or Hindu cosine could be greater than R just as no modern sine or cosine could be greater than unity.

TERMINOLOGY

N.B.—In as much this Trīpraśnādhyāya has a good number of technical terms whose understanding is necessary to understand the Hindu methods of solving diurnal problems, we shall collect here all such technical terms under this heading for guidance.

Technical term	Meaning in modern terms	Symbol if any	Formula number	Occurs under verse	Hindu formula	Modern formula
Dṛk-jyā	Hindu sine of the Zenith-distance	H sin z	9	8	H sin z	R = 3438 R sin z
Dṛgamsacāpa	Zenith-distance	Z		,,		
Digjyā*	Hindu azimuth measured from the East point	H sin a		,,	H sin a	R sin a
Krantijyā	H sine of declination	H sin δ		,,	H sin δ	R sin δ
Akshajyā	H sine of latitude	H sin φ	15	,,	H cos φ	R cos φ
Lambajyā	H sine (colatitude)	H cos φ	16	,,	H cos φ	R cos φ
Chāyā	Gnomonic shadow	S	2	,,	$\frac{\text{K H sin z}}{\text{R}}$	K sin z
Chāyākarna	Hypot. of the △ one side being S	K		,,	$\sqrt{12^2 + S^2}$	

* Azimuth is called Digamsacāpa or Digamsam.

Technical term	Meaning in modern terms	Symbol if any	Formula number	Occurs under verse	Hindu formula	Modern formula
Chāyabhuja	Perpendicular from the extremity of the shadow on the East-west line.	b	3	8	$\frac{\text{K H sin z H sin } a}{\text{R}^2}$	K sin z sin a
Chayākoti	$\sqrt{s^2 - b^2}$		7	,,	$\frac{\text{K H sin } x \text{ H cos } a}{\text{R}^2}$	K sin z cos a
Vishuvatchāya	Gnomonic shadow cast at mid-day on the equinoctial day at any place	s		,,	$\frac{12 \text{ H sin } \varphi}{\text{H cos } \varphi}$	12 tan φ
Vishuvatkarṇa	Hypot. of the △, one side being s	k		,,	$\sqrt{s^2+12^2}$	12 sec φ
Agrajyā	Hindu sine of rising azimuth	A	5	,,	$\frac{\text{R H sin } \delta}{\text{H cos } \varphi}$	$\frac{\text{R sin } \delta}{\text{Cos } \varphi}$
Karnāgrā	Agrajyā reduced from a circle of radius R to one with radius K	a	6	,,	$\frac{\text{K H sin } \delta}{\text{H cos } \varphi}$	$\frac{\text{K sin } \delta}{\text{Cos } \varphi}$
Iṣta Śanku	The Hindu cosine of Z	I. S.	8	,,	H cos z	R cos z
Iṣta Hṛti	The hypot. of the △, one side being H cos z	I. H.	11	13–17	$\frac{\text{R H cos z}}{\text{H cos } \varphi}$	$\frac{\text{R cos z}}{\text{Cos } \varphi}$

Śankutala	The thrid side of the △, above	S. T.	12	13–17	$\frac{H \cos z \; H \sin \delta}{H \cos \varphi}$	$R \cos z \tan \varphi$
Kujyā	The ⊥ar distance bet. The lines drawn parallel to the east-west line through the rising point and the point of intersection of the diurnal circle and the great circle PEω		13/	,,	$\frac{H \sin \delta \; H \sin \varphi}{H \cos \varphi}$	$R \sin \delta \tan \varphi$
Vishuvat-mandala	Celestial Equator					
Krānti Mandala	Ecliptic					
Kshitija	Horizon					
Unmandala	Great circle PEω or Equatorial horizon					
Yāmyottara-mandala	Meridian					
Samamandala	Prime-vertical					
Dṛk Mandala	Vertical					
Dhṛva Prota Vṛtta	Declination circle					

Technical term	Meaning in modern terms	Symbol if any	Formula number	Occurs under verse	Hindu formula	Modern formula
Kadamba Prota Vṛtta	Circle of celestial latitude					
Krānti	Hindu declination ie. arc of the declination circle intercepted bet. Ecliptic and Equator					
Spaṣta Krānti	Declination					
Dhṛvaka or Dhṛva	Celestial long measured from the Hindu zero point					
Sāyana Dhṛva	Celestial longitude					
Vikṣēpa	Polar latitude or the arc of the declination circle intercepted bet. the Ecliptic and the celestial body					
Sphuta Vikṣēpa	Celestial latitude					
Vishuvamsa Cāpa	Right ascension					

Antyā	The length of the $\perp^{ar}$ drawn from Q the culminating point of the celestial Equator on the line drawn parallel to the East-west line through the foot of the declination circle of the rising celestial body		14	13–17	$R+R\dfrac{H \sin \varphi\ H \sin \delta}{H \cos \varphi\ H \cos \delta}$	$R\,(1+\tan \delta \tan \varphi)$
Sama-Śanku	H cos z when the body is on the prime-vertical	S. Ś.	17	20	$\dfrac{R\ H \sin \delta}{H \sin \varphi}$	$R \sin \delta / \sin \varphi$
Taddhṛti	Perpendicular distance between the lines drawn parallel to the East-west line through the rising point and the point where the diurnal circle cuts the prime-vertical		18	,,	$\dfrac{R^2\ H \sin \varphi}{H \cos \delta\ H \sin \varphi}$	$\dfrac{R \sin \delta}{\operatorname{Cos} \varphi \sin \delta}$
Purvāparā	East-west line					
Udayāsta Sūtra	Line joining the rising and setting points					
Unmandala Śanku	H cos z when the body is on the Equatorial Horizon	U. S.	19	25	$\dfrac{H \sin \delta\ H \sin \varphi}{R}$	$R \sin \delta \sin \varphi$
Dinārdha Śanku	H cos z at the culminating point	D. S.	21	31–32	$H \cos (\varphi \underset{\sim}{+} \delta)$	$R \cos (\varphi \underset{\sim}{+} \delta)$

38

Technical term	Meaning in modern terms	Symbol if any	Formula number	Occurs under verse	Hindu formula	Modern formula
Yaṣṭi	The length of the ⊥[ar] dropped from the culminating point on a plane parallel to the Horizon and passing thro/ the point of intersection of the diurnal circle and Equatorial horizon	Y	21	33	$\frac{H \cos \varphi \; H \cos \delta}{R}$	$R \cos \varphi \cos \delta$
Hṛti	The line in the diurnal circle corresponding to Antiyā in the plane of the celestial Equator or the length of the ⊥[ar] from the culminating point on Udayāstasūtra		22	3	$H \cos \delta + \frac{H \sin \delta \; H \sin \varphi}{H \cos \varphi}$	$R(\cos\delta + \sin\varphi \tan\varphi)$
Sūtra	OM (O=centre of the sphere M=foot of ⊥[ar] on OQ from the foot of declination circle		26	53–54	$H \cos h$	$R \cos h$
Kalā	Corresponding line in diurnal circle		27	,,	$\frac{H \cos h \; H \cos \delta}{R}$	$R \cos h \cos \delta$
Iṣṭāntyā	Line corresponding to Iṣṭahṛti, on the Equatorial plane		28		$\frac{R^2 \; H \cos z}{H \cos \varphi \; H \cos \delta}$	$\frac{R \cos z}{\cos \varphi \cos \delta}$

Cara Cāpa	Arc of the celestial Equator bet. the East point and foot of the declination circle					-
Carajyā	H sine of the above or the corresponding line of Kujya in the plane of the celestial Equator		13	13–17	$\frac{R\ H\sin\varphi\ H\sin\delta}{H\cos\varphi\ H\cos\delta}$	$R\tan\varphi\tan\delta$
Natakāla	Hour angle	**h**				
Unnata	Time elapsed after rise					
Śara	Corresponding line of phala in the Equatorial plane		29	58	$H\ vers\ (h)$	$R\ (1-\cos h)$
Phala	$\perp^{ar}$ from the culminating point on a line through the celestial body parallel to Udayāstasūtra		30		$\frac{H\ vers\ h\ H\cos\delta}{R}$	$R\cos\delta\ (1-\cos h)$
Ūrdhwa	Orthogonal projection of phala on the plane of the meridian		31	59	$\frac{H\ vers\ h\ H\cos\varphi\ H\cos\delta}{R^2}$	$R\cos\varphi\cos\delta(1-\cos h)$
Iṣṭayaṣṭi	The length of the $\perp^{ar}$ dropped from the celestial body on a horizontaly plane passing through the top of U.S.	I. y	27/	55	$\frac{H\cos h\ H\cos\delta\ H\cos\varphi}{R^3}$	$R\cos\varphi\cos\delta\cos h$

Yet $\frac{R \sin \delta}{\sin \varphi}$ will have a value greater than R ie. even though the Sama-S'anku is never born so to say, it has a magnitude. 'तत्कथमिदं वन्ध्यासुतवत्' Bhāskara exclaims with respect to the magnitudes of Sama-S'anku and Taddhṛti as well, both of which are not there, yet, both of which have magnitudes greater than R. So he says "those magnitudes of the Sama-S'anku and Taddhṛti are like the sons of a barren lady". Then he says 'तद्दपि प्रदर्श्यते' ie. 'We shall show how they arise.' Here he uses his intuition of the principle of geometrical continuity. Even when the diurnal circle does not cut the prime-vertical, their planes intersect, outside the sphere and the perpendicular dropped from the point of intersection on the plane of the horizon is the Sama S'anku which has a magnitude greater than R. Similarly the Taddhṛti could be seen what it is now. These magnitudes can enter into a proportion like the I in verse 22, and do help us to get the other real magnitudes like the Unmandala S'anku etc.

Verses 66, 67 *and first half of* 68. To obtain the time from the shadow.

$$\text{Iṣṭāntyakā} = \frac{\text{U.K.} \times \text{Carajyā}}{\text{I.K.}} = \frac{\text{D.K.} \times \text{Antyā}}{\text{I.K.}}$$

$$= \frac{R^2}{\text{H} \cos \delta \times \text{I.K.}}$$

Dinārdha $- h =$ Unnatakāla

where K = Karṇa, k = Vishuvat Karṇa, I.A. = Iṣṭāntya.

Comm. We had under verse 62.

Iṣṭa-Karṇa $\times$ Iṣṭa-Hṛti = D. K. $\times$ Hṛti = S. K. $\times$ Taddhṛti = U. K. $\times$ Kujyā multiplying throughout by $\frac{R}{\text{H} \cos \delta}$

Iṣṭa-Karṇa × I. A. = D. K. × Antya = U. K. × Carajyā

$$\text{so that I.A.} = \frac{\text{D.K.} \times \text{Antyā}}{\text{I.K.}} = \frac{\text{U.K.} \times \text{Carajyā}}{\text{I.K.}} \qquad \text{I}$$

$$\text{But U.K.} = \frac{12\text{ R}}{\text{U.S.}} \text{ (verse 40)}$$

$$\therefore \text{ U.K.} \times \text{Carajyā} = \frac{12\text{R} \times \text{Carajyā}}{\text{U.S.}}$$

$$= \frac{12\text{R} \times \text{Kujyā} \times \text{R}}{\text{U.S.} \times \text{H} \cos \delta} \text{ since Carajyā} = \frac{\text{Kujyā} \times \text{R}}{\text{H} \cos \delta}$$

But Kujyā and U.S. are the Karṇa and Koṭi of the seventh latitudinal triangle so that $\frac{\text{Kujyā}}{\text{U.S.}} = \frac{k}{12}$

Comparing with the first fundamental latitudinal triangle.

$$\therefore \text{ U.K.} \times \text{Carajyā} = \frac{12\text{R}^2 \times k}{\text{H} \cos \delta \ 12} = \frac{k\text{R}^2}{\text{H} \cos \delta}$$

Hence substituting in I

$$\text{I.A.} = \frac{\text{U.K.} \times \text{Carajyā}}{\text{I.K.}} = \frac{k\text{R}^2}{\text{H} \cos \delta \times \text{I.K.}}$$

Thus we have proved the first part of the statement. Having obtained I.A., from fig. 52 we have Antyā — I.A. =(FQ—AN)=CQ. The Utkrama Cāpa of CQ = NQ = h and Dinārdha — h = Unnatakāla. Let us see what this procedure means in practice. Since on any day at any place, φ and the declination of the Sun are known we can compute all the magnitudes given in the verse or more easily H cos δ so that from the formula Iṣṭāntyā = $\frac{12\text{R}^2}{\text{H} \cos \delta \times \text{I.K.}}$ where K = $\sqrt{\text{S}^2 + 12}$, the shadow being observed Iṣṭāntya could be computed in no time. Also the Antyā of the day R+(H tan φ tan δ) can be computed so that the segment CQ can be got. The inverse Hversine of this is h. The arc CQ above was symbolized as S'ara.

Bhāskara's proof of I.A. $= \dfrac{kR^2}{H \cos \delta \times I.K.}$ proceeds from first principles as follows:—(i) If by k we have 12 as the Koṭi what have we for R? The result is H cos z, Mahās'anku. ∴ Mahā S'anku $= \dfrac{12\,R}{I.K.}$. From Maha S'anku we pass on to Iṣta-Hṛti with which it forms a latitudinal triangle. If by the gnomon of 12 units we have k the Vishuvat Karṇa, what have we by Mahā S'anku? The result is $\dfrac{12\,R}{I.K.} \times \dfrac{k}{12} = \dfrac{kR}{I.K.}$. Again from the Iṣta-Hṛti we pass on to I.A. by multiplying by $\dfrac{R}{H \cos \delta}$ so that I.A. $= \dfrac{kR}{H \cos \delta \times I.K.}$ as given.

Second half of verse 68. The inverse Hversine if a quantity x greater than R, is $5400 + H \sin^{-1}(\theta)$ when $x - R = \theta$.

Comm. Since Hvers $(90+\theta) = R + H \sin \theta = x$ (say) $90 + \theta =$ Hvers$^{-1}(R + H \sin \theta) =$

But 90° are equal to 5400 asus and $\theta = H \sin^{-1}(H \sin \theta)$ $= H \sin^{-1}(x - R) =$ त्रिस्याधि कभागस्यक्रमचापम्

∴ 5400 + त्रिज्याधिकभागक्रमचापम् = Utkrama Cāpa of a quantity.

Verse 69. Alternate method of obtaining the time that has elapsed after Sunrise noting the shadow S.

Subtract Charajyā from or add it to Iṣṭāntyā according as δ is north or south. Obtain inverse H sine of the remainder and add the Caracāpa to this inverse H sine. Then we have the unnatakāla by converting the result into time.

Comm: Ref. fig. 52. I.A. = AN. I.A. − Carajyā = B.N. $H vers^{-1}$ (BN) = arc EN. Arc EN + Caracāpa = EN + EM = MN. This converted into time is evidently the Unnatakāla because the arc MN of the equator is the arc intercepted between the feet of the declination circles at rising and at the time concerned M being the foot of the rising declination circle and N the foot of that at the time in question. The convention of signs is clear.

Verse 70 *and first half of* 71. To obtain the Sun's longitude from the shadow S.

The gnomonic shadow at noon, being multiplied by R and divided by K, the inverse H sine of the result gives the meridian zenith-distance. This being decreased or increased by the latitude gives the Sun's declination according as the extremity of the shadow is north or From the declination, we have the Sun's longitude by the formula $H \sin \delta = \frac{H \sin \lambda \; H \sin \omega}{R}$

Comm. We have from the triangle formed by the gnomon and the shadow S, $\frac{S}{K} = \sin z$ or $\frac{SR}{K} = H \sin z$

$\therefore H \sin^{-1}\left(\frac{SR}{K}\right) = z$. Since we are directed to take the mid-day shadow, we have the meridian zenith-distance and from the formula $z + \delta = \phi$ we have δ. If the extremity of the shadow be north, the Sun is south of the zenith, and then $\phi \sim z = \delta$. The word वियुक्ता: is used to signify difference which is positive. If $z > \phi$ then the declination is south and vice-versa. If the extremity of the shadow is on the south, the Sun is on the north of the zenith, in which case $\phi + z = \delta$.

Second half of verse 71. To obtain ϕ from δ.

If the zenith distance and the declination are of the same direction, their difference, otherwise their sum will be the latitude.

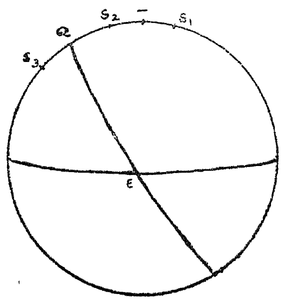


Fig. 58

Comm. Suppose the zenith-distance is north and declination also north, then clearly from fig. 58, in this position S_1 of the Sun $QS_1 - ZS_1 = \phi = \delta - Z$ (1) Again in the position S_3 of the Sun, zenith-distance is south and declination is also south; so, here also difference gives ϕ ie. $ZS_3 - QS_3 = Z - \delta = \phi$. (3) In the position S_2, however, Z is south and δ is north, so that their sum is equal to ϕ. It will be noted that the Hindu convention of signs does not contemplate negative declination and also it uses the word 'difference' to signify the positive difference, as for example, in the first two cases $\delta \sim Z$ is taken as ϕ. The modern formula $Z + \delta = \phi$ applies universally with the convention that δ is +ve if north, ϕ is +ve if south.

Verses 72, 73. To obtain the Bhuja from the shadow. Karṇa Vrittāgrā $= \frac{A \times K}{R}$ where A = Agrā and K the Chayākarṇa. This Karnāgrā is to be taken as belonging to the opposite hemisphere to the Sun. Calling this Karnāgra as a and the equinoctial shadow as s, $a \underset{\sim}{+} s$ according as δ is south or north gives the Chāyābhuja b. Thus $a \underset{\sim}{+} s = b$. If the extremity of the shadow be on the north and δ be north, then $b + a = s$; if δ be south $b \sim a = s$. If b be north, $b \sim s =$ A, otherwise ie. if south $b + s = a$. $\frac{R \times a}{K} =$ Agrā and

$$\frac{\text{Agrā} \times \text{Koṭi of latitudinal triangle}}{\text{of the latitudinal triangle}} = \quad \text{sin}$$

Comm. These verses are very important and the contents have been already elucidated under verses 13–17. We shall elucidate the convention of signs in more detail both from the modern point as well as from the Hindu traditional point. First we shall discuss the modern. We have the formula A = S + B where A = Agrā, S = S'ankutala and B, S'anku-bhuja. Let us confine ourselves to north latitudes alone, for, south latitudes did not concern the Hindu astronomers. Then treating north declination as positive and also north Hindu azimuth as positive the above formula holds universally. (Ref. fig. 59) Let the figure represent the meridian plane. Let S_1, S_2, S_3 be the projections of the Sun's positions in their diurnal circles on to the meridian plane. Let A, B. be the

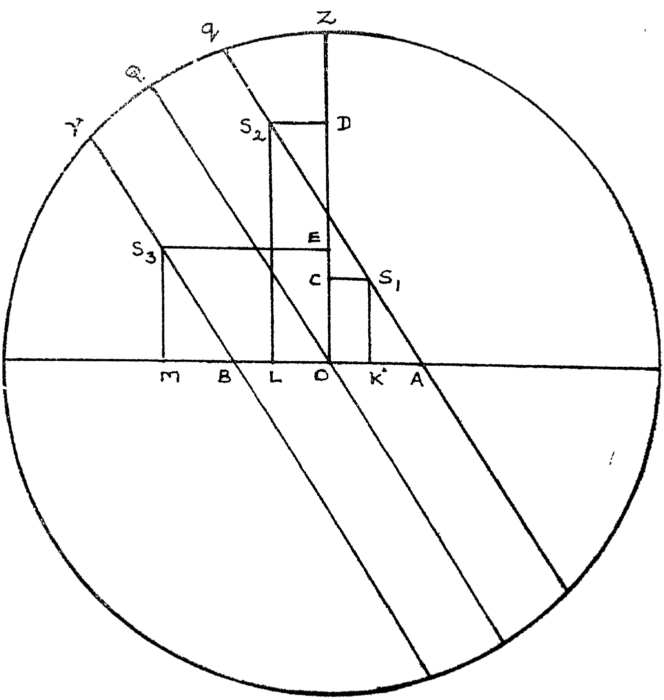


Fig. 59

projections of the rising points of the Sun on the same plane. Let perpendiculars be dropped from S_1, S_2, S_3 on the plane of the horizon. Let *o* be the centre of the sphere. Let *ns* be the north-south line. In position S_1, S_1C = S'anku-Bhuja, KA = S'ankutala, *o*A = Agrā, so that A=S+B (1) In the position S_2. S_2D = S'anku-Bhuja, LA = S'anku-tala, *o*A = Agrā, so that B + A = S ; but here B is negative, a the azimuth being south so that writing − B for B, − B + A = S $\therefore$ A = S + B again. In position S_3, OB=A, MB=S, OM=B so that A+S=B; but here A is +ve, δ being south and B is negative a being south ; hence writing − A and − B for A and B A + S = − B or A = S + B again. This shows that with the convention cited above A = S + B holds good universally.

Now let us consider the situation with respect to the Karnāgrā. Each of the three quantities *a*, *b*, *s* have now opposite directions. If δ be north, the Sun will be on the north of Equator, but the extremity of the shadow will now be on the south of the Equinoctial shadow line (E.S.L.) ie. the line which is parallel to the East-west line at a distance of the equinoctial shadow *s*, and which is the locus of the extremity of the shadow on the equinoctial day ; thus when the Agrā is considered to be positive being on the north of the East point, the Karnāgrā, though it is on the south will have to be considered positive. Similarly when the azimuth of the Sun *a* is on the north of the prime-vertical and is so considered to be positive, the extremity of the shadow being on the south of the East-west line and has a negative azimuth, the bhuja is still to be considered positive. Again the S'ankutala being always south of the Udayāsta Sūtra being considered south and positive, the corresponding quantity into which it gets converted on the horizontal dial namely the equinoctial shadow *s* will be north of the East-west line and will be considered positive. In other words in the equation $a = b + s$, a is positive when δ is north, b is +ve when the azimuth of the Sun is north of the East point, and s is always positive. Since in north latitudes, S'ankutala will be always south of the Udayāstasūtra and considered positive, the E.S.L. will be on the north of the East-west line so that s is considered positive. We should have had to consider s negative in southern latitudes, as per the above convention but as the Hindu astronomers did not have to concern themselves with south latitudes, the question of sign for s did not arise except taking it as always positive. Hence the equation $a = b + s$ will hold universally with the same conventions of sign which we stipulated with respect to the equation $A = B + S$. The foregoing analysis is on the modern lines.

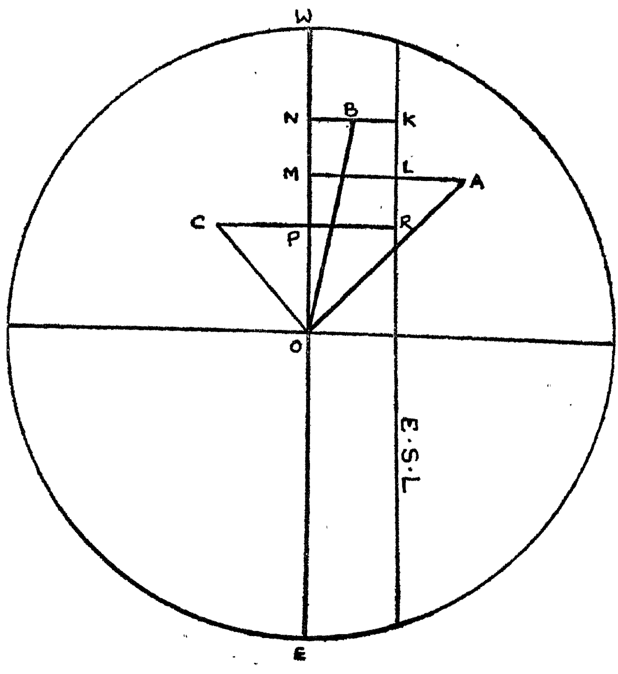


Fig. 60

Now let us see how the convention of signs is stipulated in Hindu astronomy with respect to the equation $a = b + s$. We have said that 's' is always north of the East-west line and considered positive. Regarding 'a', it is said by Bhāskara व्यस्तगोला which means that when δ is north and the Sun is said to be in northern hemisphere, a is said to belong to the southern hemisphere. Also when the Sun is on the north of the prime-vertical and S'ankubhuja is considered north, the Chāyābhuja being south of the East-west line is considered south. With these conventions of directions (we say of directions, and not signs because the Hindu astronomers do not speak of signs but only of directions) it is stipulated in Hindu astronomy that quantities of like directions are to be added, otherwise their difference is to be taken गिंगः, भिन्नदिशोः अन्तरम्'. This convention stipulating addition or difference is technically called ' Samskāra '. That is why it is said simply ie. 'Samskāra (on the aforesaid lines) is to be effected between a and s to get bhuja b'. Here it may be reiterated that the word 'अन्तरम्' ie. 'difference' is used in its restricted sense namely that the positive difference alone is to be taken. Thus the 'antaram' of 8 and 5 is 3 as well as that of 5 and 8 is also 3. Then it might be asked how to decide the direction of the bhuja, if we were to take s as equal to $a \sim b$ and not $a-b$ or $b-a$. The answer is that between a and s whose directions are known as per the aforesaid convention, equating their difference ie. $a \sim s$ to b, we have to take b as having that direction which is indicated by that quantity either a or s which has a larger numerical value.

Thus while on modern lines we take $a = b + s$ to hold universally with the convention of signs which we have agreed to on modern lines namely that a is +ve if δ is north and b is positive if the Hindu azimuth is north of the East point and s is always +ve, we take on the Hindu lines $a \underset{\sim}{+} s = b$ with the conventions stipulated with respect to directions (not of signs) namely that a is south if δ is north, s is always north and b is to be taken to belong to that direction to which the numerically bigger quantity of a and s belongs in the case of difference. Also it is to be taken to belong to that direction of a and s when both of them have the same direction.

With this convention in mind, Bhāskara clarifies the convention by citing examples.

(1) S=5, δ is north, Agrā=916′-48″ K=30 so that $a = \frac{KA}{R} = \frac{916\frac{4}{5} \times 30}{3438} = 8$ units (south, because it should be taken to be व्यस्तगोला ie. belonging to the direction opposite to that of δ).

Question. " What is b and of what direction ?"

Answer. $b = a \underset{\sim}{+} s = 8$ (south) ~ 5 (north). (We are taking the difference because Samskāra is to be construed as addition of quantities having the same direction and difference of quantities of opposite direction $\therefore$ $b = 3$ and *is on the south* because the numerically bigger quantity of 8 and 5 belongs to south.

Q. 2. δ is north S=5, A=916′-48″ ; K=15, 'what is b and in what direction ?'

Answer. $a = \frac{KA}{R} = 4$; $b = a \underset{\sim}{+} s = 4 \underset{\sim}{+} 5 = 4 \sim 5$ (here a is south δ being north and s is north so that difference is stipulated as above) $= 1$ (north because 5 belongs to north.

We add here two more examples to illustrate addition by saying that δ is south in the above examples so that a is north in both the examples. Hence in (1) $b=8+5=13$ (north) and in (2) $b=4+5=9$ (north). Refering to Fig. 60, we see there three cases depicted namely the extremities of the shadows being A, B and C. In the first case AL = Karnāgra $=a$ (Karnāgra is the distance of the extremity of the shadow from the E.S.L. namely K.L.R. whereas bhuja is the distance of the same from the East-west line namely PMN) AM = bhuja and ML $= s$ so that $b=a+s$ addition being justified since both a and s are of the same direction namely north. In the second case BK = Karnāgra, BN = bhuja and NK $= s$ so that $b=s-a$, the difference being justified because a is south and s north. Here we have taken the difference as $s-a$ and not $a-s$ because s is numerically greater and being oriented north, the bhuja is north. In the third case, PR$=s$, CP$=b$ and CR$=a$ so that $b=a-s$, the difference being justified because s is north and a south. Also we have taken the difference as $a-s$ and not $s-a$ because a is numerically greater and as such lends its direction namely 'south' to the bhuja. Thus in the three examples cited, addition is prescribed between a and s only when the extremity of the shadow is to the north of E.S.L. In the case of $A=S+B$ or $B=A-S$ also, addition is prescribed only when δ is south, which means that the corresponding a ie. AL is north. In fact the prescription of addition or difference accord in the cases of both the equations either

Verses 74 *and* 75. Hereafter questions are being set and answered on diurnal problems.

Seeing the shadow of the gnomon, the azimuth and longitude of the Sun or seeing two shadows with their respective directions, whoever knows the equinoctial shadow of the place, I consider him as the Garuda or Eagle who could overcome the false pride of puffed up snakes of astronomers.

Given that when $K=30$ units, the bhuja is 3 units south, and when $K=15$, the bhuja is 1 unit north, compute the latitude, or again given H sin δ = 846 and given K and b of a shadow, compute the equinoctial shadows.

Comm. The questions are clear the second verse illustrating the first.

Verse 76. Answer of the first question.

$\frac{b_1 K_2 \mp b_2 K_1}{K_2 \sim K_1} = s$ according as the bhujas are of the same or opposite directions.

Comm. Suppose b_1 and b_2 are of the same direction so that $b_1 = a_1 - s$ and $b_2 = a_2 - s$, taking the modern convention of signs. But $a_1 = \frac{K_1 A}{R}$ and $a_2 = \frac{K_2 A}{R}$

$$\therefore \quad b_1 = \frac{K_1 A}{R} - s \text{ and } b_2 = \frac{K_2 A}{R} - s$$

$$\therefore \quad \frac{b_1 + s}{K_1} = \frac{A}{R} = \frac{b_2 + s}{K_2}$$

$$\therefore \quad K_2 b_1 - K_1 b_2 = s (K_1 - K_2)$$

$$\therefore \quad s = \frac{K_2 b_1 - K_2 b_2}{K_1 - K_2}$$

If, however b_1 and b_2 are of opposite directions ie. of opposite signs, writing $-b_2$ for b_2, we have

$s = \frac{K_2 b_1 + K_1 b_2}{K_1 - K_2}$. Here we have chosen to follow the modern convention of signs; otherwise we have to consider four alternatives, for, bhujas of the same direction might mean both of the type OC (fig. 60) or both of the type of OB or one of the tyye OB and one of the type of OA or both of the type OA.

Verses 77 *and* 78. Answer to the second question.

Let Laghu $\equiv$ L $= \left(\frac{\text{KH} \sin \delta}{R}\right)^2$; $12^2 (L - b^2) \equiv$ Ādya; Para $= 12^2 b$. Let Ādya and Para be divided by $L \sim 12^2$; call them still Ādya and Para; then $\sqrt{\text{Para}^2 + A} \pm \text{Para} = s$ according as b is north or south.

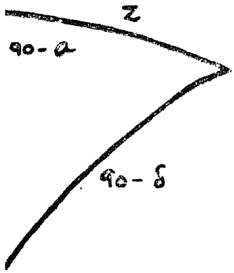


Fig. 61

Comm. The data are H sin δ, K and b; since K and b are given a is known. Thus from the triangle PZS (fig. 61) we have sin δ = sinϕ cos z+cosϕ sin z sin a, all quantities except ϕ are known. Solving this trigonometrical equation which is of the form $a \cos \phi + b \sin \phi = c$, we can have ϕ.

We shall now see how it is solved by Bhāskara. Let s be the equinoctial shadow. Then

$$a = b + s = \frac{\text{KA}}{\text{R}} = \frac{\text{KH} \sin \delta}{\text{H} \cos \phi} \quad \text{I}$$

But $\frac{12}{\text{K}} = \frac{12}{\sqrt{s^2 + 12^2}} = \frac{\text{H} \cos \phi}{\text{R}}$ so that

H cos $\phi = \frac{12 \text{ R}}{\sqrt{s^2 + 12^2}}$. Substituting in I

$$b + s = \frac{\text{KH} \sin \delta \sqrt{12^2 + s^2}}{12 \text{ R}}$$

which reduces to $12^2 \text{R}^2 (b + s)^2 = \text{K}^2 \text{H} \sin^2 \delta \, (12^2 + s^2)$

ie. $s^2 (12^2 \text{R}^2 - \text{K}^2 \text{H} \sin^2 \delta) + 2b \, 12^2 \text{R}.^2 s = 12^2 \{(\text{H} \sin^2 \delta) \text{K}^2 - b^2 \text{R}^2\}$

ie. $s^2 \left(12^2 - \frac{\text{K}^2 \text{H} \sin^2 \delta}{\text{R}^2}\right) + 2.\ 12^2\, b.\, s) = 12^2 \left(\frac{\text{K}^2 \text{H} \sin^2 \delta}{\text{R}^2} - b^2\right)$

Here $\frac{\text{K}^2 \text{H} \sin^2 \delta}{\text{R}^2}$ is symbolized as L

$12^2 \left(\frac{\text{K}^2 \text{H} \sin^2 \delta}{\text{R}^2} - b^2\right)$ put as Ādya and $12^2\, b$ is put as para.

So the equation reduces to

$s^2(12^2 - L) + 2.$ Para $s =$ Ādya ; Divide throughout by $12^2 - L$ and put again $\frac{\text{Para}}{12^2 - L}$ as Para and $\frac{\text{Adj}}{12^2 - L}$ Ādya Then the equation reduces to $s^2 + 2$ Para $s =$ Ādya ; completing the square $(s + \text{Para})^2 = \text{Para}^2 + \text{Ādya}$

$$s + \text{Para} = \sqrt{\text{Para}^2 + \text{Ādya}}$$

$$s = \sqrt{\text{Para}^2 + \text{Ādya}} - \text{Para} \text{ as one solution.}$$

We have taken to start with $a = b + s$ which holds good according to the Hindu convention when b is south ; if, however b is north $a = b \sim s$ so that $(b \sim s)^2 = s^2 + b^2 - 2bs$. So in the equation we have to write $-s$ for s, so that we have now $s^2 - 2$ Para $s =$ Ādya ie. $(s - \text{Para})^2 = \text{Para}^2 +$

$\therefore \quad s = \sqrt{\text{Para}^2 + \text{Ādya}} +$ Para as the second solution.

Verse 79. When the Sun's longitude is 135°, the shadow of the gnomon is 12 units and west. What is the latitude?

Comm. Here is a method of obtaining the latitude of the place by observing the gnomon's shadow when the Sun is on the prime-vertical.

Verse 80. Answer to the question above.

$$\frac{12\,R}{K} = H \cos z\,;$$

$$s = \frac{12\,H \sin \delta}{\sqrt{\text{Sama-S'anku}^2 - H \sin^2 \delta}}$$

Comm. Solution in modern terms.

$S = 12 \quad \therefore \tan z = 1 \quad \therefore z = 45$; but when the Sun is on the prime-vertical, we have by Napier's rule

$\sin \delta = \sin \phi \cos z = \sin \phi \sqrt{2}$. But since $\lambda = 135°$

$\text{Sin}\,\delta = \sin \lambda \sin \omega = \sin 135 \sin \omega = \sqrt{\tfrac{1}{2}} \sin \omega$

$\therefore \quad \sin \phi \sqrt{2} = \sqrt{2} \sin \omega \qquad \therefore \quad \omega = \phi$

$\therefore \ \tan \phi = \tan 24°$

$= .4452 \quad \therefore \quad s = 12 \times .4452 = 5.3424 = 5'' - 20'''$

Bhāskara's solution. Taking the fifth latitudinal triangle $\frac{\text{H} \sin \delta}{x} = \frac{s}{12}$ where x is the Koṭi

$\therefore \quad s = \frac{12 \text{ H} \sin \delta}{x}$. But we are given that the Sama-S'anku ie.

$\text{H} \cos z = \text{R} \cos z = \frac{\text{R}}{\sqrt{2}}$ because $z = 45°$ when the shadow equals the length of the gnomon.

and $\text{H} \sin \delta = \frac{\text{H} \sin \lambda \text{ H} \sin \omega}{\text{R}} = \frac{\text{H} \sin 135 \text{ H} \sin \omega}{\text{R}}$

$= \frac{\text{H} \sin 45 \text{ H} \sin \omega}{\text{R}} = \text{R}/\sqrt{2} \times \frac{\text{H} \sin \omega}{\text{R}} = \frac{\text{H} \sin \omega}{\sqrt{2}}$

$\therefore x = \sqrt{\text{Sama-S'anku}^2 - \text{H} \sin^2 \delta} = \sqrt{\frac{\text{R}^2}{2} - \frac{\text{H} \sin^2 \omega}{2}}$

$= \frac{\text{H} \cos \omega}{\sqrt{2}} \quad \therefore \quad s = \frac{12 \text{ H} \sin \delta \times \sqrt{2}}{\text{H} \cos \omega} =$

$\frac{12 \text{ H} \sin \omega}{\sqrt{2}} \times \frac{\sqrt{2}}{\text{H} \cos \omega}$

$= \frac{12 \text{ H} \sin \omega}{\text{H} \cos \omega} = \frac{12 \times 1397}{\sqrt{3438^2 - 1397^2}}$

since $\text{H} \sin \omega = 1397$ (verse 12 Spaṣṭādhikāra)

$= 5'' - 20'''$

Verse 81. Two more questions.

An observer at Ujjain observed that the Sun was on the prime-vertical 5 nādis after Sun-rise or 5 nādis afternoon. If you could give me the declination of the Sun at that moments I would reckon you as one who could be well compared with the goad that could be applied to the head of the wild elephants of puffed up astronomers.

Verses 82, 83. Answer to the first question.

Assume the H sine of the Unnatakāla to be Iṣṭa Hṛti in the first place. Multiply it by 12 s and divide by k^2 the square of the Viṣuvatkarṇa. Then you get an approximate value of H sin δ. Then compute with this, H sin δ, Charajyā etc. and thereby obtain a more correct value of the Iṣṭa Hṛti. Multiply this by the H sin δ got before and divide by the first Hṛti assumed. Then we have a nearer approximation of H sin δ. Repeat the process till a stationary value has been reached. That will be the correct H sin δ.

Comm. We know that H sine of the Unnatakāla is nearly the Iṣṭāntyakā. It will be noted that Iṣṭāntyakā is the sum of two H sines namely (1) Carajyā (2) H sine of Unnatakāla minus Chara. The second H sine is called Sūtra or H cos h. (Vide page 278).

Thus *Sūtra* + *Carajyā* = *Iṣṭāntyakā* whereas H sine (Unnatakāla) = H sine of the Cāpas of Carajyā and Sūtra. In other words H sin (Unnatakāla) = H sin (H sin^{-1} Carajyā + H sin^{-1} (Sūtra)). Iṣṭa Hṛti is Iṣṭāntyaka $\times \frac{\text{H} \cos \delta}{\text{R}}$. But we do not know H sin δ so that Iṣṭa Hṛti could not be got. So we will not be far from truth in assuming the given Unnatakāla to be Iṣṭa Hṛti itself ie. Taddhṛti here, as the Sun is on the prime-vertical. The formula for Taddhṛti is

$$\frac{}{\text{H} \cos \phi \ \text{H} \sin \phi} = \frac{\text{R}^2 . \ \text{H} \sin \delta}{\frac{\text{R} . 12}{k} \times \text{R} \times \frac{s}{k}} = \frac{\text{K}^2 \ \text{H} \sin \delta}{12 s}$$

$\therefore \quad \frac{\text{Taddhṛti} \times 12\, s}{k^2} = \sin \delta$. Thus assuming H sine of the given Unnatakāla to be Taddhṛti and multiplying it by 12 s and dividing by k^2 we have the value of H sin δ. But this is approximate because the given Unnatakāla is not exactly Taddhṛti but only an approximate value. From this H sin δ, compute H cos δ, Charajyā, Kujyā and through the process indicated in verse 54 namely "Subtract the Characāpa from the Unnatakāla. The H sine of the result is called Sūtra. Multiply the Sūtra by H cos δ and divide by R; then we have Kalā. Add Kujyā to Kalā; we get Iṣṭa-Hṛti", we obtain a more correct value of Taddhṛti. Then here we may cut short the process as follows namely 'If by the assumed Taddhṛti we had the previous H sin δ, what shall we have for this more approximate Taddhṛti'? The result will be a more approximate value of H sin δ. Again form the Taddhṛti with this H sin δ and so repeating the process till we have a stationary value, we have the correct value of H sin δ.

Note. This is a beautiful example of the method of successive approximations which is a modern technique but which was so much in vogue and favourite with the Hindu astronomers. (It will be noted how to cut short the method).

Verses 84, 85. Answer to the second question.

Obtain $12^2\, R^2/(R^2 - H \sin^2 h)\, s^2 + 1$ and divide R^2 by this and take the square root which gives H sin δ. Then $\frac{R \times H \sin \delta}{H \sin \omega}$ gives H sin λ whose Cāpa gives the longitude of the Sun.

Comm. The H sine of the given Natakāla is H sin h and $R^2 - H \sin^2 h = H \cos^2 h$. Let H sin δ be x, which is required to be found. Then $R^2 - x^2 = H \cos^2 \delta$; H cos h = Sūtra and

$$\frac{\text{Sūtra} \times \text{H} \cos \delta}{\text{R}} = \text{Kalā} = \frac{\text{H} \cos h \, \text{H} \cos \delta}{\text{R}} = \frac{\text{H} \cos h \sqrt{\text{R}^2 - x^2}}{\text{R}}$$

But Kalā is the Koti of the fifth latitudinal triangle of which H sin δ is Bhuja. Hence $\frac{\text{Kalā} \times}{\text{H} \cos} = \text{H} \sin \delta = x$

ie. $\frac{\text{H} \cos h \sqrt{\text{R}^2 - x^2}}{\text{R}} \times \frac{\text{H} \sin \phi}{\text{H} \cos \phi} = x$; but $\frac{\text{H} \sin}{\text{H} \cos \phi}$

$\therefore$ Squaring both sides $\frac{(\text{R}^2 - \text{H} \sin^2 h)(\text{R}^2 - x^2)}{\text{R}^2} \times \frac{s^2}{12^2} = x$

$\therefore \quad 12^2 \text{R}^2 x^2 = s^2 \text{R}^2 (\text{R}^2 - \text{H} \sin^2 h) - s^2 x^2 (\text{R}^2 - \text{H} \sin^2 h)$

ie. $x^2 \{(12^2 \text{R}^2 + s^2 (\text{R}^2 - \text{H} \sin^2 h)\} = s^2 \text{R}^2 (\text{R}^2 - \text{H} \sin^2 h)$

$$x^2 = \frac{s^2 \text{R}^2 (\text{R}^2 - \text{H} \sin^2 h)}{12^2 \text{R}^2 + s^2 (\text{R}^2 - \text{H} \sin^2 h)}$$

$$\frac{\text{R}^2}{\frac{12^2 \text{R}^2}{s^2 (\text{R}^2 - \text{H} \sin^2 h)} + 1} \quad \therefore x = \sqrt{\frac{\text{R}}{\frac{12^2 \text{R}^2}{s^2 (\text{R}^2 - \text{H} \sin^2 h)}}}$$

H sin δ as given. From H sin δ, the method of obtaining λ is clear from the formula $\frac{\text{H} \sin \lambda \, \text{H} \sin \omega}{\text{R}} = \text{H} \sin \delta$. In the given numerical example $h = 5$ nādis $= \frac{360°}{12} = 30°$ since 60 nadis of time correspond to 360°.

Thus H sin $h = \frac{\text{R}}{2}$; the remaining work follows.

Verse 86. Another question.

When the Sun is on the prime-vertical the gnomonic shadow is noted to be 16 inches. The Unnatakāla is 8 nādis. If you could give the H sin δ and s, I shall consider you nothing short of one who is an adept in solving the totality of the diurnal problems.

Verses 87 and 88. Answer to the question.

Here also assume H sin (Unnata) to be the Taddhṛti as formerly done. Then as the shadow is 16″,

$K = \sqrt{16^2 + 12^2} = 20''$. Then $H \cos z = \frac{12\ R}{K} = \frac{12}{20} \times 3438$

Unnatakāla = 8 nādis = 48°.

$\therefore$ H sin (48°) = assumed Taddhṛti. Then from the fourth latitudinal triangle, $\frac{H \sin 48}{H \cos z} = \frac{k}{12}$

$\therefore$ Approximate value of k is $\frac{12\ H \sin 48}{12/20 \times 3438} = \frac{20\ H \sin 48}{3438}$.

This is a known quantity from which s could be computed since $12^2 + s^2 = k^2$. Again from the fifth latitudinal triangle $\frac{s}{k} = \frac{H \sin \delta}{S.\ S.}$.

Here s, k and S. S. (Samamandala-S'anku) are known $\therefore$ H sin δ could be got approximately. Thus we have found approximately the required quantities s and H sin δ. From this H sin δ and s we have to compute again H cos δ, Carajyā, Kujyā, etc. and applying the procedure of verse 54

$H \sin \frac{(\text{Unnata-cāra}) \times H \cos \delta}{R} + \text{Kujyā} = \text{Iṣta Hṛti}$; this is nearer value of Iṣta Hṛti than the assumed Taddhṛti. From this again obtain as before s and H sin δ; we could not apply the proportion " If by the assumed Taddhṛti we have the previous H sin δ, what shall we have for this computed Taddhṛti (Iṣta Hṛti)" for the reason given below. So Repeat the entire process till an invariable quantity is got which will be the correct value of H sin δ.

Here repeating the entire calculation is correct and not taking the proportion because $\text{Taddhṛti} = \frac{R \sin \delta}{\sin \phi \cos \phi}$ where sin δ and sin φ are both to be computed.

Formerly we could take the proportion in verse 81 because the latitude of Ujjain being known, in the magnitude of Taddhṛti namely $\frac{R \text{ H sin } \delta}{\text{Sin } \phi \cos \phi}$ only H sin δ is variable and Taddhṛti is directly proportional to H sin δ. But in the pre ent example both H sin δ and H sin ϕ are both variables so that, that kind of rule of three does not work.

Verse 89. Oh! Mathematician! At a place where $s = 5''$, there 10 nādikas after Sun-rise the shadow S is observed to be $9''$. Tell me what the longitude of the Sun would be, if you are an adept in computing as well as understanding the geometry of the sphere.

Verses 90, 91. Answer to the question posed.

Assume H sine (Unnatakāla) to be Iṣṭāntyakā. Then

$$\frac{K \times \text{H cos z} \times R}{12 \times \text{I. A.}} \quad \text{H cos } \delta \text{ where I. A.} = \text{Iṣṭāntyakā.}$$

$R^2 - \text{H cos}^2 \delta = \text{H sin}^2 \delta$; from this approximate H sin δ and the given s compute a more approximate I. A. Repeat the process till an invariable quantity is obtained for H sin δ, which will be its correct value. From this, using the formula H sin $\delta = \frac{\text{H sin } \lambda \text{ H si}}{R}$

Comm. We know the formula for I. A. as

$$\frac{R^2 \text{ H cos z}}{\text{H cos } \varphi \text{ H cos } \delta}$$ Assuming Unnatakālajyā as I. A.

$$\text{H sin (Unnatakāla)} = \frac{R^2 \text{ H co}}{\text{H cos } \varphi \text{ H cos } \delta}$$

$$\frac{R^2 \text{ H cos z}}{\frac{12 \text{ R. I.A.}}{k}} = \frac{k \times R \times \text{H cos z}}{12 \times \text{I. A.}}$$; from

which obtaining H sin δ and proceeding as indicated we have λ. In the above proof we have used our formula. But Hindu Astronomers proceed from first principles. Let us hear Bhāskara. Since $S = 9''$; $K = \sqrt{9^2 + 12^2} = 15''$

$\therefore$ Mahā-S'anku $=$ H cos z $= \frac{R \times 12}{K} = \frac{3438 \times 12}{15} = 2750-24.$

We know that Mahā-S'anku forms a latitudinal triangle with Iṣta Hṛti.

So $\frac{\text{H cos z}}{\text{Iṣta Hṛti}} = \frac{12}{k}$ $\therefore$ Iṣta Hṛti $= \frac{\quad}{12}$

Iṣtāntyā $= \frac{\text{Iṣta Hṛti} \times R}{\text{H cos } \delta}$ ie, H cos $\delta = \frac{\quad}{\text{Iṣtāntyā}}$

$= \frac{\text{H cos z} \times k \times R}{12 \times \text{I. A.}}$ substituting the above value of Iṣta Hṛti. Here H cos z is got above and I. A. has been assumed above as H sin (Unnatakāla).

Note. (1) Computing

$$= \frac{12\,R}{15} \times \frac{\sqrt{s^2 + 12^2} \times}{12 \times \text{H sin } (60)}$$

$\frac{R^2 \times 13}{15 \text{ H sin } 60} = \frac{13\,R \times 2}{15 \times \sqrt{3}}$. Here H cos $\delta > R$ which is invalid.

(2) This is the only place where Bhāskara gave a numerical example with a slight flaw. In other words, under the given circumstances the shadow must be greater than what is given. However, the procedure indicated is mathematically correct.

(3) It is interesting to note that the flaw was noted by a commentator named Lakṣmidāsa as reported by Munīs'wara in his Marīchi Bhaṣya. Munīs'wara also noted the flaw but argues away in an untenable way. Another commentator named Ganes'a who was the author of the commentary named S'iromaṇipracās'a, does not seem to have noticed the flaw, or even if he did notice, probably he fought shy of pronouncing that there was a flaw. In fact a simple flaw like this in numerical examples, is not in the least derogatory to the prestige of Bhāskara. So, the commentators who happened to notice the flaw need not have pointed the same.

(4) Or again in the given place, for the value of H cos δ to be valid the Unnatakāla x must be such that

15 H sin $x > 13$ R so that H cos δ might be less than R. This means sin $x > \frac{13}{15} = .8667$ so that $x > 60°-4'$; so instead of 10 nādikas, if the time were given to be just even one Vinādika greater, it would have been alright, or again if the latitude were given to be just a little less it would have been alright.

Verse 92. Oh! Mathematician! please tell me the magnitudes of the equinoctial shadow and the longitude of the Sun if at a place on a particular day, Kujyā is 245 and Taddhṛti 3125.

Verse 93. Answer to the questiou above.

$$s = \sqrt{\frac{144 \text{ Kujyā}}{\text{Taddhṛti}-\text{Kujyā}}} \text{ and H sid } \delta = \frac{12 \text{ Kujyā}}{s}$$

and $\text{H sin } \lambda = \frac{\text{R H sin } \delta}{\text{H sin } \omega}$.

Comm. From the fifth latitudinal triangle compared with third,

$$\underset{(1)}{\frac{\text{Kujyā}}{\text{Krāntijyā}}} = \underset{(2)}{\frac{\text{Krāntijyā}}{\text{Taddhṛti}-\text{Kujyā}}} = \underset{(3)}{\frac{\text{Agrā}}{\text{S. S.}}} = \underset{(4)}{\frac{s}{12}}$$

Multiplying (1) by (2) $\frac{\text{Kujyā}}{\text{Taddhṛti}-\text{Kujyā}} = \frac{s^2}{12^2}$

$\therefore \quad s = \sqrt{\frac{144 \text{ Kujyā}}{\text{Taddhṛti}-\text{Kujyā}}}$ Also Equating (1) and (4)

$$\text{Krāntijyā} = \frac{12}{s} \text{ Kujyā}.$$

Verse 94. Given that H sin δ + S. S. + Taddhṛti—Kujyā = 6720, and Kujyā+Agrā+H sin δ = 1960. Then I shall consider him who finds s and the longitude of the Sun as the very Sun illuminating the lotuses of astronomers.

Verse 95. Answer to the question above.

Divide 12 × Second sum by the first sum, that will be s. Again $\frac{12 \times \text{Second sum}}{12 + s + k} = \text{H} \sin \delta$. From H sin δ, λ could be had as before.

Comm. Comparing the third and fifth latitudinal triangles

$$\frac{\quad}{\text{Krāntijyā}} = \frac{\text{Krāntijyā}}{\text{Taddhṛti} - \text{Kujyā}} = \frac{\text{Agrā}}{\text{S. S.}} = \frac{s}{12}$$

$$\frac{\text{Kujyā} + \text{Krāntijyā} + \text{Agrā}}{\text{Krāntijyā} + \text{Taddhṛti} + \text{S. S.} - \text{Kujyā}} = \frac{1960}{6720} = \frac{7}{24}$$

$$\therefore \quad s = 7/24 \times 12 = 7/2 = 3\tfrac{1}{2}''.$$

Again comparing the third and the first latitudinal triangles

$$\underset{(1)}{\frac{s}{\text{Kujyā}}} = \underset{(2)}{\frac{12}{\text{Krāntijyā}}} = \underset{(3)}{\frac{k}{\text{Agrā}}} = \underset{(4)}{\frac{s + 12 + k}{\text{Kujyā} + \text{Agrā} + \text{Krāntijyā}}}$$

$$= \underset{(5)}{\frac{s + 12 + k}{1960}}$$

Equating (2) and (5) $\text{Krāntijyā} = \frac{12 \times 1960}{12 + s + k}$

$$\frac{12 \times 1960}{\frac{24}{2} + \frac{25}{2}} = \frac{12 \times 1960}{28} = 840 \text{ since when } s = 7/2$$

$k = 25/2$ which is the hypotenuse of the triangle formed by the equinoctial shadow with the gnomon.

Equating (1) and (5) Kujyā = 245; equating (3) and (5) Agrā = 875.

Now from the fourth latitudinal triangle compared with the first $\underset{(1)}{\frac{\text{Agrā}}{s}} = \underset{(2)}{\frac{\text{S. S.}}{12}} = \underset{(3)}{\frac{\text{Taddhṛti}}{k}}$

From (1) and (2) S. S. $= \frac{12}{7/2} \times$ Agrā $= \frac{24}{7} \times 875 = 3000$

From (1) and (3) Taddhṛti $= \frac{k}{s} \times$ Agrā

$$= \frac{25}{9} \times \frac{2}{7} \times 875 = 3125.$$

Note. This is a beautiful example exhibiting Bhāskara's dexterity in algebra.

Verse 96. Given that the sum of H sin δ, S. S. and Taddhṛti – Kujyā = 1440, and the sum of Agrā, S. S. and Taddhṛti = 800, I shall deem him whoever finds s and the longitude of the Sun, as the very Sun illuminating the lotuses of astronomers.

Verse 97. Answer to the problem above.

The second sum divided by the first and multiplied by 12 gives k from which s could be got. Then the first sum divided by $s + 12 + k$ gives H sin δ from which the longitude of the Sun could be got.

Comm. Comparing the third and the fifth latitudinal triangles, we have

$$\underset{(1)}{\frac{\text{Agrā}}{\text{Krāntijyā}}} = \underset{(2)}{\frac{\text{S. S.}}{\text{Taddhṛti} - \text{Kujyā}}} = \underset{(3)}{\frac{\text{Taddhṛti}}{\text{S. S.}}}$$

$$= \underset{(4)}{\frac{\quad}{12}} = \underset{(5)}{\frac{\quad + \text{S. S.} + \text{Taddhṛti}}{\text{Krāntijyā} + \text{Taddhṛti} - \text{Kujyā} + \text{S. S.}}} = \frac{\quad}{1440} = \underset{(6)}{\frac{\quad}{4}}$$

Equating (4) and (6) $k = \frac{12 \times 5}{4} = 15$

$\therefore k^2 = 225 = 12^2 + s^2 \quad \therefore s = 9.$ Again comparing the fourth latitudinal triangle, with the fundamental,

$$\frac{\text{Agrā}}{s} = \frac{\text{S. S.}}{12} = \frac{\text{Taddhṛti}}{k} = \frac{\text{Agrā} + \text{S. S.} +}{s + 12 + k}$$

(1) (2) (3)

$$= \frac{1800}{9 + 12 + 15} = \frac{1800}{36} = 50 \qquad \text{II}$$

(4)

Equating (1) and (4) Agrā $= 9 \times 50 = 450$

Equating (2) and (4) S. S. $= 12 \times 50 = 600$

Thirdly Taddhṛti $= 15 \times 50 = 750$

Again Equating (1) and (6) of I

$$\frac{\text{Agrā}}{\text{Krāntijyā}} = \frac{5}{4} = \frac{450}{\text{Krāntijyā}} \quad \therefore \quad \text{H sin } \delta = \frac{450 \times 4}{5}$$

$= 360$ from which λ could be computed.

Verse 98. The chara at a place where $s = 9$, is equal to 3 nādis. If you could compute the longitude of the Sun, then certainly you are a leader among astronomers, Oh ! Scholar !

Verse 99. Answer to the problem above.

$$\frac{12 \text{ Carajyā}}{\sqrt{\left(\frac{12 \times \text{Carajyā}}{\text{R}}\right)^2 + s^2}} = \text{H sin } \delta$$ where from λ the longitude of the Sun could be computed.

Comm. Let H sin $\delta = x$; then from the third latitudinal triangle $\frac{\text{Kujyā}}{\text{Krāntijyā}} = \frac{s}{12} = \frac{9}{12} = \frac{3}{4}$

$\therefore$ Kujyā $= \frac{3x}{4}$ since Krāntijyā means H sin δ

$$\therefore \text{Carajyā} = \frac{3x}{4} \times \frac{\text{R}}{\text{H cos } \delta} = \frac{3\text{R}x}{4\sqrt{\text{R}^2 - x^2}}$$

$= \text{H sin } (3 \times 6) = \text{H sin } 18^\circ$ $\therefore$ Squaring

$$9\text{R}^2 x^2 = 16 (\text{R}^2 - x^2) \text{ H sin}^2 18 = 16 \text{ Carajyā}^2 (\text{R}^2 - x^2)$$

$\therefore \quad x^2 (9 R^2 + 16 \text{ Carajyā}^2) = 16 R^2 \text{ Carajyā}^2$

$\therefore \quad x^2 = \dfrac{16 R^2 \text{ Carajya}^2}{9 R^2 + 16 \text{ Carajyā}^2} \qquad \therefore \; x = \dfrac{4 R \text{ Carajyā}}{\sqrt{9 R^2 + 16 \text{ Carajyā}^2}}$

$= \dfrac{12 \text{ Carajyā}}{\sqrt{s^2 + \left(\dfrac{12 \text{ Carajyā}}{R}\right)^2}}$

Here Carajyā being known, H sin δ could be computed.

Verse 100. If you studied what is known as Madhyamāharaṇa, then compute λ the longitude of the Sun given that H sin δ + H cos δ + H sin λ = 5000.

Verse 101. Answer to the problem above.

Let the given sum multiplied by 4 and divided by 15 be Ādya ; then H sin δ =

$$\text{Ādya} - \sqrt{910678 - \frac{2 \text{ square of the given sum}}{337}}.$$

Comm. Let H sin $\delta = x$; then H cos δ =

$$\text{H sin } \delta = \frac{\text{H sin } \omega \text{ H sin } \lambda}{R}$$

$$= \frac{x R}{\text{H sin } \omega} = \frac{x R}{1397}$$

$\therefore$ The given sum $= x + \sqrt{R^2 - x^2} + \dfrac{x R}{1397} = 5000$

$\overline{- x^2} = 5000 - x\left(1 + \dfrac{R}{1397}\right) = 5000$ -

$\therefore \quad R^2 - x^2 = 5000^2 + x^2 \left(\dfrac{4835}{1397}\right) - \dfrac{2 \times 5000 \times 4835}{1397} x$

$\therefore \quad x^2 \left\{1 + \dfrac{4835^2}{1397^2}\right\} - \dfrac{2 \times 5000 \times 4835}{1397} = R^2 - 5000^2$